

# A thermodynamic approach to two-variable Ruelle and Selberg zeta functions via the Farey map

Claudio Bonanno <sup>\*</sup>

Stefano Isola <sup>†</sup>

## Abstract

In this paper we perform a detailed study of the spectral properties of a family of signed transfer operators  $\mathcal{P}_q^\pm$  associated to the Farey map, where  $q$  is a real or complex parameter (inverse temperature). We then extend in several directions the transfer operator approach to the Selberg zeta function for  $PSL(2, \mathbb{Z})$  introduced by Mayer. We first obtain a correspondence between the zeroes of the Selberg zeta function and the eigenfunctions of  $\mathcal{P}_q^\pm$  with eigenvalue  $\lambda = 1$ , which in particular implies that obtained by Mayer. Moreover, our approach naturally leads to study two-variable Ruelle and Selberg zeta functions,  $\zeta(q, z)$  and  $Z(q, z)$ , both of which can be written in terms of the Fredholm determinant of a two-parameter family of operators  $\mathcal{Q}_{q,z}$  associated to the Gauss map. A simple functional relation between these operators then leads to a new correspondence between zeroes and poles of these zeta functions and eigenvalues  $\lambda \in \mathbb{C} \setminus [0, 1)$ , that is outside the continuous spectrum, of  $\mathcal{P}_q^\pm$ .

**Keywords:** transfer operators; Farey map; Gauss map; Selberg zeta function; Ruelle zeta function

## 1 Introduction

The transfer operator approach to the Selberg zeta function for the full modular group  $PSL(2, \mathbb{Z})$  introduced by Mayer in [20] led to new interesting interactions between number theory and the thermodynamic approach to dynamical systems. The established correspondence between the zeroes of the Selberg zeta function and the eigenfunctions of the transfer operators for the Gauss map has been subsequently studied in particular in connection with the theory of period functions for cusp and non-cusp forms on  $PSL(2, \mathbb{Z})$  in [6, 7, 17]. In this paper we extend the approach in [20] to signed transfer operators  $\mathcal{P}_q^\pm$  for the Farey map, which is connected to the Gauss map by an induction procedure. This approach clarifies some aspects of the results in [20], and we are naturally led to the definitions of two-variable Ruelle and Selberg zeta functions for which extensions of Mayer results hold.

We now discuss in details the contents of the paper. The first issue is the study of the signed transfer operators  $\mathcal{P}_q^\pm$  for the Farey map defined in (2.1). The spectral properties of transfer operators for uniformly expanding maps of an interval are now well understood and depend crucially on the Banach space considered [3]. For sufficiently regular functions it turns out that the transfer operator is quasi-compact, hence the spectrum is made of isolated eigenvalues with finite multiplicity

---

<sup>\*</sup>(corresponding author) Dipartimento di Matematica Applicata, Università di Pisa, via F. Buonarroti 1/c, I-56127 Pisa, Italy, email: <bonanno@mail.dm.unipi.it>

<sup>†</sup>Dipartimento di Matematica e Informatica, Università di Camerino, via Madonna delle Carceri, I-62032 Camerino, Italy. e-mail: <stefano.isola@unicam.it>

and the essential part, a disk of radius strictly smaller than the spectral radius. However the essential spectral radius depends on the expanding constant  $\rho$  of the map and on the degree of regularity of the functions. In particular as  $\rho \rightarrow 1$  from above, the essential spectral radius converges to the spectral radius. The Farey map  $F$  is a prototype of smooth intermittent map on the unit interval  $[0, 1]$ , being expanding everywhere except that at the origin, a neutral fixed point. Hence for the Farey map  $\rho = 1$ , and classical approaches to the spectral properties of its transfer operator fail. As a matter of fact, using *ad hoc* techniques, the spectrum of the Farey transfer operator when acting on suitable spaces of holomorphic functions has been shown to have an absolutely continuous component given by the unit interval (see [23, 13, 5]).

Here we consider the family of signed generalised transfer operators  $\mathcal{P}_q^\pm$  associated to  $F$  and defined in (2.4) providing a detailed study of their spectral properties on a space of holomorphic functions on an open domain containing  $(0, 1)$ . In Section 2, we prove our first main result which is a complete characterisation of the eigenfunctions of  $\mathcal{P}_q^\pm$  with eigenvalues not embedded in the continuous spectrum (Theorem 2.8). This is obtained in terms of an integral transform defined on the weighted spaces  $L^p((0, +\infty), m_q(t))$  with  $dm_q(t) = t^{2q-1}e^{-t}$ . A similar approach has been used in [13, 21, 11, 5]. In particular in [5] the Hilbert space  $L^2((0, +\infty), m_q(t))$  has been studied in some detail and the resulting issues are used in this paper. In particular, a simple argument (Proposition 2.1) shows that eigenfunctions of  $\mathcal{P}_q^\pm$  satisfy the three-term functional equation (2.6), which for  $\lambda = 1$  is but the Lewis functional equation studied in [17], where a class of solutions of this equation is proved to be in one-to-one correspondence with the Maass cuspidal and non-cuspidal forms on the full modular group  $PSL(2, \mathbb{Z})$ . Some of our results, in particular Corollary 2.10, are then generalisations of results in [17] to eigenvalues  $\lambda \neq 1$ .

In the second part, we use an inducing procedure for  $F$  which was first introduced in [22] for a general class of intermittent interval maps. The idea is to consider an induced map  $G$  on the interval with respect to the first passage time at a subset of  $[0, 1]$  away from the neutral fixed point, and to study the spectral properties of the transfer operators  $\mathcal{Q}$  of  $G$ . Then functional relations between  $\mathcal{Q}$  and  $\mathcal{P}$  allow to translate the spectral properties of  $\mathcal{Q}$  into those of  $\mathcal{P}$  and viceversa. For applications of this method see also [23, 13, 21]. It is well known that the Farey map is related to number theory, and in particular to the continued fractions expansion of  $x \in [0, 1]$ . Moreover, if we define  $G$  to be the induced map with respect to the first passage time at the interval  $(\frac{1}{2}, 1]$ , it turns out that  $G$  is the Gauss map on the unit interval, see (3.1). In Section 3, following this procedure, we define in (3.4) a two-parameter family of transfer operators  $\mathcal{Q}_{q,z}$  associated to  $G$  and look for functional relations between  $\mathcal{Q}_{q,z}$  and  $\mathcal{P}_q^\pm$ . This is obtained in Theorem 3.6 and allows us to obtain a one-to-one correspondence between all eigenfunctions to the eigenvalues  $\pm 1$  of  $\mathcal{Q}_{q,z}$  and some eigenfunctions with eigenvalue  $\frac{1}{z}$  of  $\mathcal{P}_q^\pm$  (Corollary 3.7).

Let us recall that the properties of the transfer operators  $\mathcal{Q}_{q,1}$  have been already studied in [18, 19, 20, 7], and also discussed in [17]. In particular, in [20] it is proved that the eigenfunctions with eigenvalues  $\pm 1$  of  $\mathcal{Q}_{q,1}$  are in one-to-one correspondence with the zeroes of the Selberg zeta function for the full modular group  $PSL(2, \mathbb{Z})$ . In turn, it is the content of Selberg trace formula that the zeroes of the Selberg zeta function are in one-to-one correspondence with the Maass cusp forms and the non-trivial zeroes of the Riemann zeta function, which are related to the Maass non-cuspidal forms. One then obtains a relation between eigenfunctions to the eigenvalues  $\pm 1$  of  $\mathcal{Q}_{q,1}$  and a class of solutions of the Lewis three-term functional equation. This is discussed in [17] and proved without appealing to the Selberg zeta function.

It is one of the aims of this paper to obtain the latter relation from the eigenfunctions of the

transfer operators  $\mathcal{P}_q^\pm$ , thus giving, in the same spirit of [20], a thermodynamic formalism approach to the period functions studied in [17]. The first step is Theorem 3.6 mentioned above. Notice in particular the term  $\pm c\mu^x$  in (3.9). This term is indeed responsible for the restriction to a class of eigenfunctions of  $\mathcal{P}_q^\pm$  in Corollary 3.7, i.e. to a class of solutions of the Lewis functional equation. The second step is the main result of Section 4, Theorem 4.6. We first show that the transfer operators  $\mathcal{Q}_{q,z}$  are of the trace class on a suitable Banach space, hence the Fredholm determinants  $\det(1 \mp \mathcal{Q}_{q,z})$  are well defined, and the traces can be computed by the same computations as in [18] and [13]. Hence we obtain (4.7), which is contained in [20] for  $z = 1$ , and leads to a definition of a two-variable Selberg zeta function  $Z(q, z)$ . Altogether these steps show that the zeroes of the function  $Z(q, z)$  are in one-to-one correspondence with the eigenfunctions of  $\mathcal{P}_q^\pm$  with eigenvalue  $\frac{1}{z}$  such that the constant  $c$  which appears in (3.9) (see also Corollary 2.10) vanishes. In the case  $z = 1$ , as a corollary of our results we also obtain a new proof for the characterisation of the zeroes of the function  $Z(q, 1)$  in terms of even and odd cusp and non-cusp forms given in [9]. This is the content of Theorem 4.8 in Section 4.1.

In the same Section we also discuss a two-variable Ruelle zeta function  $\zeta(q, z)$  for the Farey map, defined in (4.2). Our main result is equation (4.8), which gives an expression of  $\zeta(q, z)$  again in terms of the Fredholm determinants  $\det(1 \mp \mathcal{Q}_{q,z})$ , extending a previous result in [13]. For a discussion of multi-variable zeta functions in dynamical systems and number theory see [15].

Finally in Section 5 we come back to the Farey map and its relations to number theory. Using a formal manipulation of (4.7) and (4.8), which is similar to the approach in [23], we obtain an expression for the zeta functions  $Z(q, z)$  and  $\zeta(q, z)$  as exponentials of power series whose coefficients are obtained as sums along lines of the Farey tree (Theorem 5.3).

## 2 Transfer operators for the Farey map

Let  $F : [0, 1] \rightarrow [0, 1]$  be the *Farey map* defined by

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \quad (2.1)$$

To this map we can associate a family of *signed generalized transfer operators*  $\mathcal{P}_q^\pm$ , for a complex parameter  $q$  with  $\operatorname{Re}(q) > 0$ , whose action on a function  $f(x) : [0, 1] \rightarrow \mathbb{C}$  is given by a weighted sum over the values of  $f$  on the set  $F^{-1}(x)$ , namely letting

$$(\mathcal{P}_{0,q}f)(x) := \left(\frac{1}{x+1}\right)^{2q} f\left(\frac{x}{x+1}\right) \quad (2.2)$$

$$(\mathcal{P}_{1,q}f)(x) := \left(\frac{1}{x+1}\right)^{2q} f\left(\frac{1}{x+1}\right) \quad (2.3)$$

we define

$$(\mathcal{P}_q^\pm f)(x) := (\mathcal{P}_{0,q}f)(x) \pm (\mathcal{P}_{1,q}f)(x) \quad (2.4)$$

Since the operators  $\mathcal{P}_q^\pm$  are defined by multiplication and composition by real analytic maps which extend to holomorphic maps on a neighbourhood of the interval  $[0, 1]$ , it is natural to consider the action of  $\mathcal{P}_q^\pm$  on the space  $\mathcal{H}(B)$  of holomorphic functions on the open domain

$$B := \left\{ x \in \mathbb{C} : \left| x - \frac{1}{2} \right| < \frac{1}{2} \right\}.$$

We point out that we still denote by  $x$  the complex variable. We are interested in the spectral properties of the transfer operators, hence first of all we give a characterisation of the eigenfunctions. To this aim we give some properties of the action of  $\mathcal{P}_q^\pm$  on  $\mathcal{H}(B)$ . It is useful to consider also the operator  $\mathcal{J}_q$  which is the involution acting as

$$(\mathcal{J}_q f)(x) := \frac{1}{x^{2q}} f\left(\frac{1}{x}\right) \quad (2.5)$$

By definition of involution, any function  $f \in \mathcal{H}(\{\operatorname{Re}(x) > 0\})$  can be written as a sum of eigenfunctions of  $\mathcal{J}_q$ , that is  $f = \varphi_+ + \varphi_-$  with  $\mathcal{J}_q \varphi_+ = \varphi_+$  and  $\mathcal{J}_q \varphi_- = -\varphi_-$ .

**Proposition 2.1.** (i) If  $f \in \mathcal{H}(B)$  then  $\mathcal{P}_q^\pm f \in \mathcal{H}(\{\operatorname{Re}(x) > 0\})$ ;  
(ii) if  $f \in \mathcal{H}(B)$  is an eigenfunction of  $\mathcal{P}_q^\pm$  with eigenvalue  $\lambda \in \mathbb{C} \setminus \{0\}$ , then  $f \in \mathcal{H}(\{\operatorname{Re}(x) > 0\})$  and  $\mathcal{J}_q f = \pm f$ . In particular if  $\mathcal{J}_q f = -f$  then  $f(1) = 0$ . Moreover

$$\lambda f(x) - f(x+1) = \left(\frac{1}{x+1}\right)^{2q} f\left(\frac{x}{x+1}\right) \quad \forall \operatorname{Re}(x) > 0; \quad (2.6)$$

(iii) if  $f \in \mathcal{H}(\{\operatorname{Re}(x) > 0\})$  satisfies (2.6) with  $\lambda \in \mathbb{C} \setminus \{0\}$ , then  $\varphi_+ := \frac{1}{2}(f + \mathcal{J}_q f)$  satisfies  $\mathcal{P}_q^+ \varphi_+ = \lambda \varphi_+$  and  $\varphi_- := \frac{1}{2}(f - \mathcal{J}_q f)$  satisfies  $\mathcal{P}_q^- \varphi_- = \lambda \varphi_-$ .

*Proof.* (i) follows simply by the fact that if  $\operatorname{Re}(x) > 0$  then both  $\frac{x}{x+1}$  and  $\frac{1}{x+1}$  are in  $B$ .

(ii) The first assertion follows by (i). For the second, by the definition of  $\mathcal{J}_q$  given in (2.5) one easily checks that

$$\mathcal{J}_q \mathcal{P}_q^\pm f = \pm \mathcal{P}_q^\pm f.$$

Hence if  $f$  is an eigenfunction we can rewrite the previous expression substituting  $\mathcal{P}_q^\pm f$  with  $\lambda f$  and the second assertion follows. Finally (2.6) is obtained by using the fact that for eigenfunctions it holds

$$\pm(\mathcal{P}_{1,q} f)(x) = \pm(\mathcal{J}_q f)(x+1) = f(x+1).$$

(iii) Let  $\varphi_+$  and  $\varphi_-$  be defined as above. They satisfy  $f = \varphi_+ + \varphi_-$  and  $\mathcal{J}_q \varphi_\pm = \pm \varphi_\pm$ . Moreover, one can easily check that if  $f$  satisfies (2.6) with  $\lambda \in \mathbb{C} \setminus \{0\}$  then the same holds true for  $\mathcal{J}_q f$ . Hence also  $\varphi_\pm$  satisfy (2.6) with the same  $\lambda$ , and their invariance properties under  $\mathcal{J}_q$  imply the assertion.  $\square$

*Remark 2.2.* Eq. (2.6) has been studied in [16, 17] for  $\lambda = 1$  in connection with Maass forms on the full modular group  $PSL(2, \mathbb{Z})$ . Moreover part (iii) of the proposition implies that eq. (2.6) has no solutions for  $|\lambda|$  bigger than both the spectral radii of  $\mathcal{P}_q^\pm$ .

We now introduce a family of spaces  $\mathcal{H}_{q,\mu}$  to which the eigenfunctions of  $\mathcal{P}_q^\pm$  belong. Throughout this paper we are assuming  $\operatorname{Re}(q) > 0$ . Let  $\mathcal{L}$  and  $\mathcal{L}^{-1}$  denote the Laplace and inverse Laplace transform, respectively. We recall that

$$\mathcal{L}(t_+^{\nu-1}) = \Gamma(\nu) x^{-\nu} \quad \mathcal{L}^{-1}(x^{-\nu}) = \frac{t_+^{\nu-1}}{\Gamma(\nu)} \quad t \in \mathbb{R}, \nu \in \mathbb{C} \setminus \mathbb{Z}^- \quad (2.7)$$

where  $\Gamma(\nu)$  is the usual Gamma function,  $t_+^\alpha := t^\alpha H(t)$ , being  $H(t)$  the Heaviside function, and for  $\text{Re}(\nu) < 0$  we are working with generalised functions. Moreover, we consider the integral transform  $\mathcal{B}_q$  introduced in [13] and defined as

$$\phi \mapsto \mathcal{B}_q[\phi](x) := \frac{1}{x^{2q}} \int_0^\infty e^{-\frac{t}{x}} e^t \phi(t) m_q(dt) \quad (2.8)$$

where  $m_q$  is the absolutely continuous measure on  $\mathbb{R}^+$  defined as  $m_q(dt) = t^{2q-1} e^{-t} dt$ . It is immediate to check that

$$L^1(m_q) \ni \phi \mapsto \mathcal{B}_q[\phi] \in \mathcal{H}(B)$$

Moreover, since  $m_q(0, \infty) = \Gamma(2q)$ , it holds  $L^p(m_q) \subset L^1(m_q)$  for all  $p \in [1, \infty]$ . We now study the spaces which contain the eigenfunctions of  $\mathcal{P}_q^\pm$  in terms of the  $\mathcal{B}_q$  transform.

**Definition 2.3.** Let  $\mathcal{H}_{q,\mu}^p$ , with  $\text{Re}(q) > 0$ ,  $q \neq \frac{1}{2}$ ,  $\mu \in \mathbb{C}$  and  $p \in [1, +\infty]$ , be the space of functions  $f(x)$  which on  $B = \{|x - \frac{1}{2}| < \frac{1}{2}\}$  can be written as

$$f(x) = \frac{c \mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q \left[ \frac{b}{t} + \phi(t) \right]$$

for  $c, b \in \mathbb{C}$  and  $\phi$  a function in  $L^p(m_q)$ , where  $m_q(dt) = t^{2q-1} e^{-t} dt$ .

*Remarks 2.4.* We give some considerations on the definition of the spaces  $\mathcal{H}_{q,\mu}^p$ . First of all to prove that Definition 2.3 is well posed, we extend the definition of the  $\mathcal{B}_q$  transform by means of (2.7) to let

$$\mathcal{B}_q \left[ \frac{1}{t} \right] = \frac{1}{x^{2q}} \int_0^\infty e^{-\frac{t}{x}} \frac{1}{t} t^{2q-1} dt = \frac{1}{x^{2q}} \mathcal{L}(t^{2q-2})|_{\frac{1}{x}} = \Gamma(2q-1) x^{-1} \quad (2.9)$$

which is well defined for  $\text{Re}(q) > 0$  and  $q \neq \frac{1}{2}$ . Notice that instead the first integral is defined only for  $\text{Re}(q) > \frac{1}{2}$ . Hence, the condition  $q \neq \frac{1}{2}$  only depends on the presence of the term  $\frac{1}{t}$ . Moreover, in this paper a typical example of functions to which we apply the  $\mathcal{B}_q$  transform is

$$\frac{b}{t} + \phi(t) = \frac{e^{-t}}{1 - e^{-t}} \frac{a_0}{\Gamma(2q)} + \frac{e^{-t}}{1 - e^{-t}} \sum_{n \geq 1} \frac{a_n}{\Gamma(n+2q)} t^n \quad t \in \mathbb{R}^+ \quad (2.10)$$

with  $\limsup_n (a_n)^{\frac{1}{n}} \leq 1$ . In this case,  $b = \frac{a_0}{\Gamma(2q)}$  and  $\phi$  is in  $L^2(m_q)$  with  $\phi(0) = \frac{a_1}{\Gamma(2q+1)} - \frac{a_0}{2\Gamma(2q)}$ . A further consideration is about the term to which we apply  $\mathcal{B}_q$ . This term could be written in many different ways, and actually some of them will be used below. However the main feature of this term that we want to stress is that it can be divided into two parts with respect to the behaviour at the origin: the first part has a singularity at  $t = 0^+$  of order  $t^{-1}$ , hence in particular does not belong to  $L^p(m_q)$  for  $\text{Re}(q) \leq \frac{p}{2}$ ; the second part is instead in  $L^p(m_q)$ .

**Proposition 2.5.** *The transfer operators  $\mathcal{P}_q^\pm$  leave invariant the spaces  $\mathcal{H}_{q,\mu}^2$  for any  $\mu \in \mathbb{C}$ . In particular*

$$\begin{aligned} \mathcal{P}_{0,q} \left( \frac{c \mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q[\psi] \right) &= \frac{c \mu \mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q[M(\psi)] \\ \mathcal{P}_{1,q} \left( \frac{c \mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q[\psi] \right) &= \mathcal{B}_q[c \bar{\phi}_{q,\mu} + N_q(\psi)] \end{aligned} \quad (2.11)$$

where  $M$  and  $N_q$  are linear operators defined by

$$M(\psi)(t) = e^{-t} \psi(t) \quad (2.12)$$

$$N_q(\psi)(t) = \int_0^\infty \frac{J_{2q-1}(2\sqrt{st})}{(st)^{q-\frac{1}{2}}} \psi(s) m_q(ds) \quad (2.13)$$

being  $J_p$  the Bessel function of order  $p$ , and the function  $\bar{\phi}_{q,\mu}(t)$  is defined by

$$\bar{\phi}_{q,\mu}(t) := \mu \sum_{k=0}^{\infty} \frac{(\log \mu)^k}{\Gamma(k+1)\Gamma(k+2q)} t^k \quad (2.14)$$

letting  $\bar{\phi}_{q,0}(t) \equiv 0$ .

*Proof.* To prove (2.11), we study the action of the transfer operators on the different terms of a function  $f \in \mathcal{H}_{q,\mu}^2$ , for  $q$  and  $\mu$  fixed.

First of all, it is immediate that  $\mathcal{P}_{0,q}\left(\frac{\mu^{\frac{1}{x}}}{x^{2q}}\right) = \mu^{\frac{1}{x^{2q}}}$  and  $\mathcal{P}_{1,q}\left(\frac{\mu^{\frac{1}{x}}}{x^{2q}}\right) = \mu^{x+1}$ . It follows that the function  $\bar{\phi}_{q,\mu}$  we are looking for has to satisfy

$$\mu^{x+1} = \mathcal{B}_q[\bar{\phi}_{q,\mu}] = \frac{1}{x^{2q}} \mathcal{L}(t_+^{2q-1} \bar{\phi}_{q,\mu}(t))|_{\frac{1}{x}}$$

hence

$$\bar{\phi}_{q,\mu}(t) = \frac{1}{t_+^{2q-1}} \mathcal{L}^{-1}\left(\frac{\mu^{1+\frac{1}{x}}}{x^{2q}}\right)$$

Expression (2.14) follows by a straightforward computation. It remains to prove that the function in the right hand side of (2.14) admits a  $\mathcal{B}_q$  transform. If we differentiate term by term, it follows that for any  $n \in \mathbb{N}$  there exists a positive polynomial  $p_n(t)$  of degree  $n$  such that

$$\frac{d}{dt} \bar{\phi}_{q,\mu}(t) \leq p_n(t) + \frac{|\log \mu|}{n+2q} |\bar{\phi}_{q,\mu}(t)|$$

From this it follows that the behaviour of  $\bar{\phi}_{q,\mu}$  at infinity is slower than  $e^{\varepsilon t}$  for all  $\varepsilon > 0$ . Moreover  $\bar{\phi}_{q,\mu}(0) = \frac{\mu}{\Gamma(2q)}$ . Hence actually  $\bar{\phi}_{q,\mu} \in L^2(m_q)$ .

For  $f \in \mathcal{H}_{q,\mu}^2$  we write  $f(x) = \frac{c\mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q[\psi]$ , where  $\psi(t) = \frac{b}{t} + \phi(t)$ . Expressions (2.11) have been proved in [8] for functions  $\phi$  in  $L^2(m_q)$ . It remains to prove that the same holds for the term  $\frac{b}{t}$ . For  $\mathcal{P}_{0,q}$  we simply have

$$\mathcal{P}_{0,q} \mathcal{B}_q \left[ \frac{1}{t} \right] = \frac{1}{x^{2q}} \int_0^\infty e^{-\frac{t}{x}} \frac{e^{-t}}{t} t^{2q-1} dt = \mathcal{B}_q \left[ \frac{e^{-t}}{t} \right]$$

For  $\mathcal{P}_{1,q}$  we have to prove

$$\mathcal{P}_{1,q} \left( \mathcal{B}_q \left[ \frac{1}{t} \right] \right) = \mathcal{B}_q \left[ N_q \left( \frac{1}{t} \right) \right]$$

Using the power series expansion

$$J_p(x) = \frac{x^p}{2^p} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(m+p+1)}$$

for the Bessel function  $J_{2q-1}$ , we have

$$\begin{aligned} N_q \left( \frac{1}{t} \right) &= \int_0^\infty \sum_{m=0}^\infty \frac{(-1)^m t^m}{m! \Gamma(m+2q)} s^{m+2q-2} e^{-s} ds = \sum_{m=0}^\infty \frac{(-1)^m t^m}{m! \Gamma(m+2q)} \mathcal{L}(s^{m+2q-2})|_{x=1} = \\ &= \sum_{m=0}^\infty \frac{(-1)^m \Gamma(m+2q-1)}{m! \Gamma(m+2q)} t^m \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{B}_q \left[ N_q \left( \frac{1}{t} \right) \right] &= \frac{1}{x^{2q}} \mathcal{L} \left( t_+^{2q-1} N_q \left( \frac{1}{t} \right) \right) \Big|_{\frac{1}{x}} = \frac{1}{x^{2q}} \sum_{m=0}^\infty \frac{(-1)^m \Gamma(m+2q-1)}{m! \Gamma(m+2q)} \mathcal{L} \left( t_+^{m+2q-1} \right) \Big|_{\frac{1}{x}} = \\ &= \sum_{m=0}^\infty \frac{(-1)^m \Gamma(m+2q-1)}{m!} x^m = \Gamma(2q-1) (1+x)^{1-2q} \end{aligned}$$

Moreover,

$$\mathcal{P}_{1,q} \left( \mathcal{B}_q \left[ \frac{1}{t} \right] \right) = \int_0^\infty e^{-tx} e^{-t} t^{2q-2} dt = \mathcal{L}(e^{-t} t^{2q-2}) = \Gamma(2q-1) (1+x)^{1-2q}$$

This completes the proof.  $\square$

*Remark 2.6.* Notice that the  $\mathcal{P}_{1,q}(\mathcal{H}_{q,\mu}^2)$  is contained in the set of functions which are  $\mathcal{B}_q$  transform of functions in  $L^2(m_q)$ . This follows from the fact that  $\bar{\phi}_{q,\mu} \in L^2(m_q)$ ,  $N_q(\phi) \in L^2(m_q)$  for any  $\phi \in L^2(m_q)$  (see [5, 8]), and writing

$$N_q \left( \frac{1}{t} \right) = \sum_{m=0}^\infty \frac{(-1)^m}{m! (m+2q-1)} t^m = \frac{\int_0^t s^{2q-2} e^{-s} ds}{t^{2q-1}} = \frac{\Gamma(2q-1) - \int_t^\infty s^{2q-2} e^{-s} ds}{t^{2q-1}} \quad (2.15)$$

where the third term makes sense only for  $\operatorname{Re}(q) > \frac{1}{2}$  and the last one for all  $\operatorname{Re}(q) > 0$  and  $t > 0$ . In particular it follows from (2.15) that  $N_q \left( \frac{1}{t} \right)$  is bounded as  $t \rightarrow 0^+$ , and by the last term it is  $O(t^{1-2q})$  as  $t \rightarrow +\infty$ . Moreover using generalized functions we can write

$$\frac{\mu^{\frac{1}{x}}}{x^{2q}} = \mathcal{B}_q \left[ \frac{\delta(-\log \mu)}{t^{2q-1}} \right] \quad \frac{\mu \mu^{\frac{1}{x}}}{x^{2q}} = \mathcal{B}_q \left[ M \left( \frac{\delta(-\log \mu)}{t^{2q-1}} \right) \right]$$

and

$$\bar{\phi}_{q,\mu}(t) = \mu \frac{J_{2q-1}(2\sqrt{-t \log \mu})}{(\sqrt{-t \log \mu})^{q-\frac{1}{2}}} = N_q \left( \frac{\delta(-\log \mu)}{t^{2q-1}} \right)$$

where  $J_p$  denotes the Bessel function. Hence given the spaces

$$\mathcal{K}_{q,\mu}^2 := \left\{ \chi(t) = c \frac{\delta(-\log \mu)}{t^{2q-1}} + \frac{b}{t} + \phi(t) : c, b \in \mathbb{C}, \phi \in L^2(m_q) \right\}$$

it holds

$$\mathcal{H}_{q,\mu}^2 = \mathcal{B}_q[\mathcal{K}_{q,\mu}^2]$$

and moreover

$$\begin{aligned}\mathcal{P}_{0,q} \left( \frac{c\mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q[\psi] \right) &= \mathcal{B}_q \left[ M \left( c \frac{\delta(-\log \mu)}{t^{2q-1}} + \psi \right) \right] \\ \mathcal{P}_{1,q} \left( \frac{c\mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q[\psi] \right) &= \mathcal{B}_q \left[ N_q \left( c \frac{\delta(-\log \mu)}{t^{2q-1}} + \psi \right) \right]\end{aligned}$$

that is for any  $\chi \in \mathcal{K}_{q,\mu}^2$

$$\mathcal{P}_{0,q}(\mathcal{B}_q[\chi]) = \mathcal{B}_q[M(\chi)] \quad \mathcal{P}_{1,q}(\mathcal{B}_q[\chi]) = \mathcal{B}_q[N_q(\chi)]$$

The spaces  $\mathcal{K}_{q,\mu}^2$  can be written as

$$\mathcal{K}_{q,\mu}^2 = \text{Span}_{\mathbb{C}} \left( \frac{\delta(-\log \mu)}{t^{2q-1}} \right) \oplus \text{Span}_{\mathbb{C}} \left( \frac{1}{t} \right) \oplus L^2(m_q)$$

*Remark 2.7.* Notice that (2.11) holds also on  $\mathcal{H}_{q,\mu}^1$  for  $\mathcal{P}_{0,q}$  but not for  $\mathcal{P}_{1,q}$ . In fact  $N_q(\phi)$  is not defined for all functions  $\phi \in L^1(m_q)$ .

**Theorem 2.8.** *If  $f \in \mathcal{H}(B)$  and  $\mathcal{P}_q^\pm f = \lambda f$  with  $\lambda \in \mathbb{C} \setminus [0, 1)$  then  $f(x) \in \mathcal{H}_{q,\lambda}^2$ .*

*Proof.* We use the properties of the inverse Laplace transform. In particular, we recall that

**Lemma 2.9.** *A function  $u(x)$  is the Laplace transform of a generalised function if and only if there exists  $k \in \mathbb{R}$  such that  $u(x)$  is holomorphic in the half-plane  $\{\text{Re}(x) > k\}$  and*

$$|u(x)| \leq M(1 + |x|)^m \quad \text{Re}(x) > k$$

for given constants  $M, m$ .

Let now  $f \in \mathcal{H}(B)$  satisfy  $\mathcal{P}_q^\pm f = \lambda f$  for some  $\lambda \in \mathbb{C} \setminus [0, 1)$ . We study separately the cases  $\lambda = 1$  and  $\lambda \in \mathbb{C} \setminus [0, 1]$ . Let us first consider the case  $\lambda = 1$ . By Proposition 2.1.(ii), the function  $f$  satisfies eq. (2.6), and we can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \text{for } \{|x-1| < 1\} \quad (2.16)$$

where the sequence  $\{a_n\}$  satisfies  $\limsup_n (a_n)^{\frac{1}{n}} \leq 1$ , and the convergence is uniform on any compact set contained in  $\{|x-1| < 1\}$ . By Lemma 2.9, the right-hand side of eq. (2.6) is the Laplace transform of a generalised function. By using (2.7) and (2.16) and easy properties of the Laplace transform, we obtain

$$\mathcal{L}^{-1} \left( \left( \frac{1}{x+1} \right)^{2q} f \left( \frac{x}{x+1} \right) \right) = t_+^{2q-1} e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{\Gamma(n+2q)} t^n \quad (2.17)$$

The left hand-side of eq. (2.6) is then the Laplace transform of a generalised function. Moreover, from eq. (2.6) with  $\lambda = 1$  we obtain

$$f(x+n) - f(x) = \sum_{h=0}^{n-1} (f(x+h+1) - f(x+h)) = - \sum_{h=0}^{n-1} \frac{f \left( \frac{x+h}{x+h+1} \right)}{(x+h+1)^{2q}}$$



Hence  $f$  satisfies the assumptions of Lemma 2.9 for any  $k > 0$ . Hence

$$\mathcal{L}^{-1}(f(x) - f(x+1)) = (1 - e^{-t}) (\mathcal{L}^{-1}f)(t) \quad (2.18)$$

Putting together (2.17) and (2.18), we obtain that (2.6) implies that there exists a constant  $c \in \mathbb{C}$  such that

$$f(x) = c + \mathcal{L} \left( \frac{t_+^{2q-1} e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)} \right) \quad (2.19)$$

Moreover, since by Proposition 2.1.(ii) it holds  $\mathcal{J}_q f = \pm f$ , we can apply the operator  $\mathcal{J}_q$  to the right-hand side of (2.19) to have  $\pm f$ . It is a straightforward computation that applying  $\mathcal{J}_q$  to the Laplace transform gives the  $\mathcal{B}_q$  transform defined in (2.8), hence

$$f(x) = \pm \left( \frac{c}{x^{2q}} + \mathcal{B}_q \left[ \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)} \right] \right) \quad (2.20)$$

and the thesis follows for  $\lambda = 1$ , with  $\mu = 1$ ,  $b = \pm \frac{a_0}{\Gamma(2q)}$  and  $\phi(t)$  as in (2.10). Let us now consider the case  $\lambda \in \mathbb{C} \setminus [0, 1]$ . Let us first assume that  $f$  is bounded at  $x = 0$ . In this case,  $\mathcal{J}_q f = \pm f$  implies that  $|f(x)| = \mathcal{O}(|x|^{-2q})$  as  $\text{Re}(x) \rightarrow \infty$ , hence  $f$  satisfies the assumptions of Lemma 2.9 with  $k = 0$ . In particular, using again the expansion (2.16), we can write eq. (2.17) and the analogous of (2.18) which is

$$\mathcal{L}^{-1}(\lambda f(x) - f(x+1)) = (\lambda - e^{-t}) (\mathcal{L}^{-1}f)(t)$$

Hence in this case we get

$$f(x) = \mathcal{L} \left( \frac{t_+^{2q-1} e^{-t}}{\lambda - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)} \right) \quad (2.21)$$

By using again the same argument as before applying the involution  $\mathcal{J}_q$  to the right-hand side of (2.21), it follows that

$$f(x) = \mathcal{B}_q \left[ \frac{e^{-t}}{\lambda - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)} \right] \quad (2.22)$$

hence  $f \in \mathcal{H}_{q,\lambda}^2$  with  $c = b = 0$ , and

$$\phi(t) = \frac{e^{-t}}{\lambda - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)} \quad (2.23)$$

which can be written in the form (2.10) with  $\phi(0) = \frac{1}{\lambda-1} \frac{a_0}{\Gamma(2q)}$ . We finish the proof by studying the case  $\lambda \in \mathbb{C} \setminus [0, 1]$  when the function  $f(x)$  is not bounded at  $x = 0$ . Let us consider the function  $g(x) := \frac{f(x)}{\lambda^x}$ . By eq. (2.6) we can write

$$f(x+n) = \lambda^n f(x) - \sum_{h=0}^{n-1} \lambda^{(n-h-1)} \frac{f\left(\frac{x+h}{x+h+1}\right)}{(x+h+1)^{2q}}$$

which implies that  $g$  satisfies

$$g(x+n) = g(x) - \sum_{h=0}^{n-1} \lambda^{-(x+h+1)} \frac{f\left(\frac{x+h}{x+h+1}\right)}{(x+h+1)^{2q}}$$

Hence depending on whether  $|\lambda| \leq 1$  or  $|\lambda| > 1$ , either  $f$  or  $g$  satisfy the assumptions of Lemma 2.9 for any  $k > 0$ . Since eq. (2.17) still applies, if  $|\lambda| \leq 1$  we can repeat the same argument as above and obtain (2.21). If  $|\lambda| > 1$ , eq. (2.6) implies that  $g$  satisfies

$$g(x) - g(x+1) = \frac{1}{\lambda^{x+1}} \left( \frac{1}{x+1} \right)^{2q} f\left(\frac{x}{x+1}\right) = \sum_{n=0}^{\infty} (-1)^n a_n \frac{e^{-(x+1)\log \lambda}}{(x+1)^{n+2q}} \quad (2.24)$$

where we have again used the expansion (2.16) for  $f$ . Both the right-hand side of (2.24) and  $g$  satisfy the assumptions of Lemma 2.9. Hence we can apply  $\mathcal{L}^{-1}$  to both sides of (2.24) and write the analogous of (2.18) for  $g$ , to obtain that there exists  $c \in \mathbb{C}$  such that

$$g(x) = c + \mathcal{L} \left( \frac{(t - \log \lambda)_+^{2q-1} e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{\Gamma(n+2q)} (t - \log \lambda)^n \right) \quad (2.25)$$

where we have used the notation  $(t - \log \lambda)_+ := (t - \log |\lambda|)_+ - i \arg \lambda$ , and the relation

$$\mathcal{L}^{-1} \left( \frac{e^{-(x+1)\log \lambda}}{(x+1)^{n+2q}} \right) = \frac{e^{-t} (t - \log \lambda)_+^{n+2q-1}}{\Gamma(n+2q)}$$

Now, since  $\mathcal{J}_q f = \pm f$ , we obtain from (2.25) and the definition of (2.8)

$$f(x) = \pm \frac{\lambda^{\frac{1}{x}}}{x^{2q}} g\left(\frac{1}{x}\right) = \pm \left( \frac{c \lambda^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q \left[ \frac{e^{-t}}{\lambda - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)} \right] \right) \quad (2.26)$$

and the thesis follows with  $b = 0$  and  $\phi$  as in (2.23).  $\square$

**Corollary 2.10.** *The eigenfunctions  $f$  of  $\mathcal{P}_q^{\pm}$  with eigenvalue  $\lambda \in \mathbb{C} \setminus [0, 1)$  have the form*

$$f(x) = \frac{c \lambda^{\frac{1}{x}}}{x^{2q}} + \frac{\Gamma(2q-1)}{\Gamma(2q)} \frac{b}{x} + \mathcal{B}_q[\phi] \quad (2.27)$$

with  $c, b \in \mathbb{C}$  and  $\phi \in L^2(m_q)$  with  $\phi(0)$  finite and  $\phi(t) = \phi(0) + O(t)$  as  $t \rightarrow 0^+$ . Moreover if  $\lambda \neq 1$  then  $b = 0$ . Instead if  $\lambda = 1$  then  $b = f(1)$ , and  $b = f(1) = 0$  if  $f$  is an eigenfunction of  $\mathcal{P}_q^-$ . Finally the function  $\phi \in L^2(m_q)$  is such that  $\mathcal{B}_q[\phi]$  is bounded as  $x \rightarrow 0$ .

*Proof.* The form (2.27) follows from the proof of Theorem 2.8 (see (2.20), (2.22), (2.26)) and (2.9). That  $b = 0$  if  $\lambda \neq 1$  follows from (2.22) and (2.26), and that  $b = f(1)$  if  $\lambda = 1$  follows from (2.20). That  $f(1) = 0$  for eigenfunctions of  $\mathcal{P}_q^-$  for any  $\lambda$  follows from Proposition 2.1-(ii). We now have to prove boundedness of  $\mathcal{B}_q[\phi]$  as  $x \rightarrow 0$ . We start with the case  $\lambda = 1$ . From eq. (2.20), it follows that we have to prove boundedness of  $\mathcal{B}_q[\psi]$  with

$$\psi(t) := \frac{e^{-t}}{1 - e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)} - \frac{a_0}{\Gamma(2q)} \frac{1}{t} \quad (2.28)$$

Since for any  $\phi \in L^2(m_q)$  the definition of the  $\mathcal{B}_q$  transform implies

$$\mathcal{B}_q[\phi] \left( \frac{1}{x} \right) = x^{2q} \mathcal{L} \left( t_+^{2q-1} \phi \right) (x)$$

we need to show that  $x^{2q} \mathcal{L} \left( t_+^{2q-1} \psi \right) (x)$  is bounded as  $x \rightarrow \infty$ . Since  $\limsup_n (a_n)^{\frac{1}{n}} \leq 1$ , the power series in (2.28) converges uniformly for  $t \in \mathbb{R}$ . Hence it holds

$$\mathcal{L} \left( t_+^{2q-1} \psi \right) = \sum_{n=1}^{\infty} \frac{(-1)^n a_n}{\Gamma(n+2q)} \mathcal{L} \left( \frac{t_+^{n+2q-1} e^{-t}}{1 - e^{-t}} \right) + \frac{a_0}{\Gamma(2q)} \mathcal{L} \left( \frac{t_+^{2q-1} e^{-t}}{1 - e^{-t}} - t_+^{2q-2} \right) \quad (2.29)$$

Moreover, since  $e^{-t} < 1$  for  $t > 0$ , for all  $n \geq 1$  we get for  $\text{Re}(q) > 0$

$$\mathcal{L} \left( \frac{t_+^{n+2q-1} e^{-t}}{1 - e^{-t}} \right) = \sum_{k=1}^{\infty} \mathcal{L} \left( e^{-kt} t_+^{n+2q-1} \right) = \sum_{k=1}^{\infty} \frac{\Gamma(n+2q)}{(x+k)^{n+2q}} = \Gamma(n+2q) \zeta_H(n+2q, x+1) \quad (2.30)$$

where  $\zeta_H(s, a)$  denotes the Hurwitz zeta function, and we have used uniform convergence of the series of functions in the first equality. The same argument works for  $n = 0$  if  $\text{Re}(q) > \frac{1}{2}$ , hence using analytical continuation of  $\zeta_H(2q, x+1)$  to  $\text{Re}(q) > \frac{1}{2}$ ,  $q \neq \frac{1}{2}$ , we write

$$\mathcal{L} \left( \frac{t_+^{2q-1} e^{-t}}{1 - e^{-t}} - t_+^{2q-2} \right) = \Gamma(2q) \zeta_H(2q, x+1) - \Gamma(2q-1) \frac{1}{x^{2q-1}} \quad (2.31)$$

Putting together (2.29), (2.30) and (2.31), we get

$$x^{2q} \mathcal{L} \left( t_+^{2q-1} \psi \right) = x^{2q} \sum_{n=0}^{\infty} (-1)^n a_n \zeta_H(n+2q, x+1) - a_0 \frac{\Gamma(2q-1)}{\Gamma(2q)} \frac{x^{2q}}{x^{2q-1}} \quad (2.32)$$

To study the behaviour of (2.32) as  $x \rightarrow \infty$ , we use the formula (see [1, pag.269]) valid for  $\text{Re}(n+2q) > 0, n+2q \neq 1$

$$\zeta_H(n+2q, x+1) = \frac{1}{(x+1)^{n+2q}} + \frac{(x+1)^{1-n-2q}}{n+2q-1} - (n+2q) \int_0^{\infty} \frac{t - [t]}{(t+x+1)^{n+2q+1}} dt \quad (2.33)$$

Using (2.33), we write (2.32) as a sum of different pieces. First, we isolate the  $n = 0$  term and the last term in (2.32), which give

$$\begin{aligned} & a_0 \frac{x^{2q}}{(x+1)^{2q}} + \frac{a_0}{2q-1} \frac{x^{2q}}{(x+1)^{2q}} (x+1) - \frac{a_0}{2q-1} x - a_0 2q x^{2q} \int_0^{\infty} \frac{t - [t]}{(t+x+1)^{2q+1}} dt = \\ & = O(1) + \frac{a_0}{2q-1} \left( \frac{x^{2q}}{(x+1)^{2q}} - 1 \right) x + \frac{a_0}{2q-1} \frac{x^{2q}}{(x+1)^{2q}} + O(1) = O(1) \end{aligned}$$

as  $x \rightarrow \infty$ , using

$$x^{2q} \left| \int_0^{\infty} \frac{t - [t]}{(t+x+1)^{2q+1}} dt \right| \leq x^{2q} \int_0^{\infty} \frac{1}{(t+x+1)^{2q+1}} dt = \frac{1}{2q} \frac{x^{2q}}{(x+1)^{2q}}$$

Now we come to the terms in (2.32) with  $n \geq 1$ . From (2.33) we get

$$\begin{aligned} & \left| x^{2q} \sum_{n=1}^{\infty} (-1)^n a_n \zeta_H(n+2q, x+1) \right| \leq \\ & \leq \left| \frac{x^{2q}}{(x+1)^{2q}} \right| \sum_{n=1}^{\infty} \frac{|a_n|}{(x+1)^n} + \left| \frac{x^{2q}}{(x+1)^{2q}} \right| \sum_{n=1}^{\infty} \frac{|a_n|}{|n+2q-1|(x+1)^{n-1}} + \\ & + |x^{2q}| \sum_{n=1}^{\infty} |a_n| |n+2q| \int_0^{\infty} \frac{1}{|(t+x+1)^{n+2q+1}|} dt \end{aligned}$$

and the right hand side is  $O(1)$  as  $x \rightarrow \infty$ . Hence the thesis follows for  $\lambda = 1$ . For  $\lambda \neq 1$ , boundedness of  $\mathcal{B}_q[\phi]$  follows from (2.22) and (2.26). Indeed in eq. (2.22), the function  $f(x)$  is bounded as  $x \rightarrow 0$  by assumption, and  $f(x) = \mathcal{B}_q[\phi]$ . Moreover from eq. (2.26) it follows that the function  $\phi$  is the same as in (2.22), hence  $\mathcal{B}_q[\phi]$  is again bounded as  $x \rightarrow 0$ .  $\square$

An example of eigenfunctions of  $\mathcal{P}_q^+$  with  $\lambda = 1$  is the family of functions defined as

$$f_q^+(x) = \frac{\zeta_R(2q)}{2} \left( 1 + \frac{1}{x^{2q}} \right) + \sum_{m,n \geq 1} \frac{1}{(mx+n)^{2q}} \quad \text{for } \operatorname{Re}(q) > 1$$

where  $\zeta_R(s)$  denotes the Riemann zeta-function. It is shown in [17] that the family  $f_q^+$  can be analytically continued to  $q \in \mathbb{C}$ . In particular  $f_1^+(x) = \frac{1}{x}$  is the invariant density of the Farey map and the only eigenfunction of  $\mathcal{P}_1^+$  with  $\lambda = 1$ . We can write the functions  $f_q^+$  as in (2.27). First of all, notice that

$$\begin{aligned} & \sum_{m,n \geq 1} \frac{1}{(mx+n)^{2q}} = \frac{1}{x^{2q}} \sum_{m,n \geq 1} \frac{1}{n^{2q} \left( \frac{m}{n} + \frac{1}{x} \right)^{2q}} = \frac{1}{\Gamma(2q) x^{2q}} \sum_{m,n \geq 1} \frac{1}{n^{2q}} \mathcal{L} \left( t_+^{2q-1} e^{-\frac{m}{n}t} \right) \left( \frac{1}{x} \right) = \\ & = \frac{1}{\Gamma(2q)} \sum_{m,n \geq 1} \frac{1}{n^{2q}} \mathcal{B}_q \left[ e^{-\frac{m}{n}t} \right] = \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)} \sum_{m,n \geq 1} \frac{e^{-\frac{m}{n}t}}{n^{2q}} \right] = \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)} \sum_{n \geq 1} \frac{1}{n^{2q}} \frac{e^{-\frac{t}{n}}}{1 - e^{-\frac{t}{n}}} \right] = \\ & = \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)t} \sum_{k \geq 0} \frac{B_k}{k!} t^k \left( \sum_{n \geq 1} \frac{1}{n^{k+2q-1}} \right) \right] = \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)t} \sum_{k \geq 0} \frac{B_k \zeta_R(k+2q-1)}{k!} t^k \right] \end{aligned}$$

where  $\{B_k\}$  are the Bernoulli numbers, and the series converges for  $|t| < 2\pi$  since  $\zeta_R(k+2q-1) = O(1)$  as  $k \rightarrow \infty$ . Hence we get

$$f_q^+(x) = \frac{\zeta_R(2q)}{2} \frac{1}{x^{2q}} + \frac{\Gamma(2q-1)}{\Gamma(2q)} \frac{\zeta_R(2q-1)}{x} + \mathcal{B}_q[\phi] \quad (2.34)$$

where  $\phi$  is for  $|t| < 2\pi$

$$\phi(t) = \frac{\zeta_R(2q)}{2\Gamma(2q)} + \frac{1}{\Gamma(2q)} \sum_{k \geq 1} \frac{B_k \zeta_R(k+2q-1)}{k!} t^{k-1}$$

Moreover in [17] it is also proved that any eigenfunction of  $\mathcal{P}_q^+$  with  $\lambda = 1$ , when written as in (2.27), satisfies  $c = \alpha \frac{1}{2} \zeta_R(2q)$  and  $b = \alpha \zeta_R(2q - 1)$  for some  $\alpha \in \mathbb{C}$  (see [17, Remark 1, pag.246]). An example of eigenfunctions of  $\mathcal{P}_q^-$  with  $\lambda = 1$  is the family of functions

$$f_q^-(x) = 1 - \frac{1}{x^{2q}} = -\frac{1}{x^{2q}} + \mathcal{B}_q \left[ \frac{1}{\Gamma(2q)} \right]$$

in particular  $b = 0$  as stated in Corollary 2.10. Moreover, any other eigenfunction of  $\mathcal{P}_q^-$  with  $\lambda = 1$  can be written as in (2.27) with  $c = b = 0$ . Indeed,  $b = 0$  since it is an eigenfunction of  $\mathcal{P}_q^-$  as stated in Corollary 2.10, and  $c$  can be eliminated by subtracting a multiple of  $f_q^-$ .

### 3 Induced operators

The Farey map  $F$  has at least two induced versions. The first one is the well-known *Gauss map*  $G$  which is obtained by iterating  $F$  once plus the number of times necessary to reach the interval  $[1/2, 1]$ . The map  $G : [0, 1] \rightarrow [0, 1]$  is given by

$$G(x) = \begin{cases} F^{[1/x]}(x) = \left\{ \frac{1}{x} \right\} & x > 0 \\ 0 & x = 0 \end{cases} \quad (3.1)$$

It is easily checked that the Gauss map acts on the continued fraction expansion of a number  $x \in [0, 1]$  as:  $x = [a_1, a_2, a_3, \dots]$  implies  $G(x) = [a_2, a_3, \dots]$ . We now introduce a family of operators related to the Gauss map. Consider the spaces

$$\mathcal{H}_q^p := \{g \in \mathcal{H}_{q,\mu}^p : c = b = 0\} \subset \mathcal{H}(B), \quad \text{Re}(q) > 0 \text{ and } p \in [1, +\infty] \quad (3.2)$$

A function  $g$  in  $\mathcal{H}_q^p$  is a  $\mathcal{B}_q$  transform of an  $L^p(m_q)$  function. Moreover, let

$$\tilde{\mathcal{H}}_q^p := \{g \in \mathcal{H}_q^p : g = \mathcal{B}_q[\phi] \text{ with } \phi(t) = \phi(0) + O(t^\varepsilon) \text{ for some } \varepsilon > 0 \text{ as } t \rightarrow 0^+\} \quad (3.3)$$

We introduce the family of operators

$$\mathcal{Q}_{q,z} : \mathcal{H}_q^1 \rightarrow \mathcal{H}(B), \quad z \in \mathbb{C} \setminus [1, +\infty) \text{ and } \text{Re}(q) > 0$$

$$\mathcal{Q}_{q,1} : \tilde{\mathcal{H}}_q^1 \rightarrow \mathcal{H}(B), \quad \text{Re}(q) > \frac{1}{2}$$

defined by

$$g(x) = \mathcal{B}_q[\phi(t)] \longmapsto (\mathcal{Q}_{q,z}g)(x) = z \mathcal{P}_{1,q} \mathcal{B}_q [(1 - z e^{-t})^{-1} \phi(t)] \quad (3.4)$$

The operators are well-defined since  $(1 - z e^{-t})^{-1} \phi(t)$  is in  $L^1(m_q)$  for any  $\phi$  in  $L^1(m_q)$ , being  $(1 - z e^{-t})^{-1}$  in  $L^\infty$  for  $z \in \mathbb{C} \setminus [1, +\infty)$ . If instead  $z = 1$  then  $(1 - e^{-t})^{-1} \phi(t)$  is in  $L^1(m_q)$  if  $\phi(0)$  is finite since  $\frac{1}{t}$  is in  $L^1(m_q)$  for  $\text{Re}(q) > \frac{1}{2}$ .

*Remark 3.1.* Notice that  $\mathcal{Q}_{q,z}(\mathcal{H}_q^p)$  is contained in  $\mathcal{H}_q^2$  for all  $p \geq 2$ , since by Remark 2.6 the set  $\mathcal{P}_{1,q}(\mathcal{H}_q^2)$  is in  $\mathcal{H}_q^2$ . Hence, using Proposition 2.5, the operators  $\mathcal{Q}_{q,z}$  induce the operators

$$Q_{q,z} : L^2(m_q) \rightarrow L^2(m_q)$$

given by

$$\phi \longmapsto Q_{q,z}\phi = z N_q (1 - z M)^{-1} \phi \quad (3.5)$$

Notice that for  $z = 1$ , the condition  $Q_{q,1}\phi = \phi$  is identical to the integral equation studied by Lewis in [16].

**Theorem 3.2.** *The operators  $\mathcal{Q}_{q,z}$  admit the integral representation*

$$(\mathcal{Q}_{q,z} \mathcal{B}_q[\phi(t)])(x) = z \int_0^\infty \frac{e^{-t(x+1)} t^{2q-1}}{(1 - z e^{-t})} \phi(t) dt \in \mathcal{H}(B) \quad (3.6)$$

*In particular for  $\operatorname{Re}(q) > 0$  the operator-valued function  $z \mapsto \mathcal{Q}_{q,z}$  is analytic in  $\mathbb{C} \setminus [1, +\infty)$ . Moreover, for  $z \in \mathbb{C} \setminus [1, +\infty)$ , the operator-valued function  $q \mapsto \mathcal{Q}_{q,z}$  is analytic for  $\operatorname{Re}(q) > 0$ , and the operator-valued function  $q \mapsto \mathcal{Q}_{q,1}$  can be extended to a meromorphic function for  $\operatorname{Re}(q) > 0$  with a simple pole at  $q = \frac{1}{2}$  with residue the operator  $g = \mathcal{B}_q[\phi] \mapsto (\mathcal{R}_{\frac{1}{2}} g)(x) = \phi(0)$ .*

*Proof.* The integral representation (3.6) is a straightforward consequence of definitions of  $\mathcal{B}_q$  in (2.8) and  $\mathcal{P}_{1,q}$  in (2.3). Moreover, since

$$|e^{-t(x+1)}| \leq e^{-t} \quad \forall x \in B$$

and  $(1 - z e^{-t})^{-1} \phi(t)$  is in  $L^1(m_q)$ , with  $m_q(dt) = e^{-t} t^{2q-1} dt$  the integral in (3.6) is finite. The statements on analyticity of  $z \mapsto \mathcal{Q}_{q,z}$  on  $z \in \mathbb{C} \setminus [1, +\infty)$  for  $\operatorname{Re}(q) > 0$ , and of  $q \mapsto \mathcal{Q}_{q,z}$  on  $\operatorname{Re}(q) > 0$  for  $z \in \mathbb{C} \setminus [1, +\infty)$ , follow by the uniform convergence of the integral representation.

For the same reason, the operator-valued function  $q \mapsto \mathcal{Q}_{q,1}$  is analytic on  $\operatorname{Re}(q) > \frac{1}{2}$ . Moreover, if  $\phi \in L^1(m_q)$  and  $\phi(t) = \phi(0) + O(t^\varepsilon)$  for some  $\varepsilon > 0$  as  $t \rightarrow 0^+$ , then we can write  $\frac{\phi(t)}{(1 - e^{-t})} = \frac{b}{t} + \bar{\phi}(t)$ , with  $b = \phi(0)$  and  $\bar{\phi} \in L^1(m_q)$  for all  $\operatorname{Re}(q) > 0$ . Hence, writing

$$\int_0^\infty \frac{e^{-t(x+1)} t^{2q-1}}{(1 - e^{-t})} \phi(t) dt = \int_0^\infty \frac{b e^{-t(x+1)} t^{2q-1}}{t} dt + \int_0^\infty e^{-t(x+1)} t^{2q-1} \bar{\phi}(t) dt \quad (3.7)$$

the meromorphic continuation of  $\mathcal{Q}_{q,1}$  to  $\operatorname{Re}(q) > 0$  is obtained by using the identity

$$\int_0^\infty \frac{b e^{-t(x+1)} t^{2q-1}}{t} dt = \mathcal{L} \left( b e^{-t} t_+^{2q-2} \right) = \frac{b \Gamma(2q-1)}{(x+1)^{2q-1}}$$

which has a simple pole only at  $q = \frac{1}{2}$  with residue  $b$ . The second term in (3.7) is analytic on  $\operatorname{Re}(q) > 0$ . Hence the thesis follows by recalling that  $b = \phi(0)$ .  $\square$

*Remark 3.3.* Notice that for  $\phi(t) = 1$ , which is in  $L^1(m_q)$  for  $\operatorname{Re}(q) > 0$ , we have

$$(\mathcal{Q}_{q,z} \mathcal{B}_q[1])(x) = z \Gamma(2q) \Phi(z, 2q, x+1)$$

where  $\Phi(z, s, a)$  is the Lerch zeta function, defined for  $|z| < 1$  and  $\operatorname{Re}(a) > 0$  as

$$\Phi(z, s, a) = \sum_{k=0}^\infty \frac{z^k}{(k+a)^s}$$

(see e.g. [10, vol.II, pag.27]).

**Theorem 3.4.** *The functions  $\mathcal{Q}_{q,z} g$ , with  $g(x)$  bounded at  $x = 0$ , admit the power series expansion*

$$(\mathcal{Q}_{q,z} g)(x) = \sum_{n \geq 1} \frac{z^n}{(x+n)^{2q}} g \left( \frac{1}{x+n} \right) \quad (3.8)$$

*for  $|z| < 1$  if  $\operatorname{Re}(q) > 0$ , and for  $|z| \leq 1$  if  $\operatorname{Re}(q) > \frac{1}{2}$ .*

*Proof.* Using definition (3.4), simply write (when  $\sum_{n=0}^{\infty} z^{n+1} e^{-t(x+1)} t^{2q-1} e^{-nt} \phi(t)$  is uniformly convergent, hence for any  $g$  fixed for  $\text{Re}(q)$  large enough, and hence for analytic continuation in  $q$  for any  $\text{Re}(q)$  for which the series makes sense)

$$\begin{aligned}
(\mathcal{Q}_{q,z}g)(x) &= z\mathcal{P}_{1,q} \left( \sum_{n=0}^{\infty} z^n \mathcal{B}_q [e^{-nt} \phi(t)] \right) \\
&= \sum_{n=0}^{\infty} z^{n+1} \int_0^{\infty} e^{-t(x+1)} t^{2q-1} e^{-nt} \phi(t) dt \\
&= \sum_{n=0}^{\infty} \frac{z^{n+1}}{(x+n+1)^{2q}} \mathcal{B}_q [\phi] \left( \frac{1}{x+n+1} \right) \\
&= \sum_{n \geq 1} \frac{z^n}{(x+n)^{2q}} g \left( \frac{1}{x+n} \right)
\end{aligned}$$

The conditions for convergence are immediate.  $\square$

*Remark 3.5.* From (3.8) it follows that the operators  $\mathcal{Q}_{q,1}$  coincide with the generalised transfer operators of the Gauss map  $G$  for functions  $g$  bounded at  $x = 0$ .

We now study the relations between  $\mathcal{Q}_{q,z}$  and  $\mathcal{P}_q^{\pm}$ .

**Theorem 3.6.** *Let  $f \in \mathcal{H}_{q,\mu}^1$  with  $z = \frac{1}{\mu} \in \mathbb{C} \setminus (1, \infty)$ . Then if  $\text{Re}(q) > \frac{1}{2}$  it holds*

$$(1 \mp \mathcal{Q}_{q,z}) (1 - z \mathcal{P}_{0,q}) f = (1 - z \mathcal{P}_q^{\pm}) f \pm c \mu^x \quad (3.9)$$

*If  $\text{Re}(q) \leq \frac{1}{2}$ ,  $q \neq \frac{1}{2}$ , then equality (3.9) holds for  $f \in \mathcal{H}_{q,1}^1$ , and if  $\mu \neq 1$  for functions  $f \in \mathcal{H}_{q,\mu}^1$  with  $b = 0$ .*

*Proof.* First of all, we show that the left-hand side of (3.9) is well defined. This follows using Proposition 2.5 and Remark 2.7, and writing

$$\begin{aligned}
(1 - z \mathcal{P}_{0,q}) f &= (1 - z \mathcal{P}_{0,q}) \left( \frac{c \mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q \left[ \frac{b}{t} + \phi(t) \right] \right) \\
&= \frac{c \mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q \left[ \frac{b}{t} + \phi(t) \right] - z \left( \frac{c \mu \mu^{\frac{1}{x}}}{x^{2q}} + \mathcal{B}_q \left[ M \left( \frac{b}{t} + \phi(t) \right) \right] \right) \\
&= \mathcal{B}_q \left[ \frac{b}{t} (1 - z) \right] + \mathcal{B}_q \left[ \phi(t) (1 - z e^{-t}) + z \frac{b(1 - e^{-t})}{t} \right]
\end{aligned}$$

The last term is in  $\mathcal{H}_q^1$  since the function in square brackets is in  $L^1(m_q)$  for any  $\phi \in L^1(m_q)$  and for all  $\text{Re}(q) > 0$ . Whereas the first term is in  $L^1(m_q)$  for  $\text{Re}(q) > \frac{1}{2}$ , and for  $\text{Re}(q) \leq \frac{1}{2}$  vanishes for  $\mu = 1$ , and for  $\mu \neq 1$  we have  $b = 0$ . Hence we can now apply  $(1 - \mathcal{Q}_{q,z})$ . Let first  $\mu \neq 1$  and  $\text{Re}(q) \leq \frac{1}{2}$ , then it follows

$$\begin{aligned}
(1 \mp \mathcal{Q}_{q,z}) (1 - z \mathcal{P}_{0,q}) f &= (1 \mp \mathcal{Q}_{q,z}) \mathcal{B}_q [\phi(t) (1 - z e^{-t})] \\
&= \mathcal{B}_q [\phi(t) (1 - z e^{-t})] - z \mathcal{P}_{1,q} \mathcal{B}_q \left[ (1 - z e^{-t})^{-1} (\phi(t) (1 - z e^{-t})) \right] \\
&= (1 - z \mathcal{P}_{0,q} - z \mathcal{P}_{1,q}) \mathcal{B}_q [\phi] = (1 - z \mathcal{P}_q^+) f
\end{aligned}$$

where we have used again Proposition 2.5 and Remark 2.7. The case  $\mu \neq 1$  and  $\operatorname{Re}(q) > \frac{1}{2}$  follows in the same way, by writing  $f = \mathcal{B}_q[\tilde{\phi}]$  with  $\tilde{\phi} = \frac{b}{t} + \phi \in L^1(m_q)$ . In the case  $\mu = 1$  instead we get

$$\begin{aligned}
(1 \mp \mathcal{Q}_{q,1}) (1 - \mathcal{P}_{0,q}) f &= (1 \mp \mathcal{Q}_{q,1}) \mathcal{B}_q \left[ \phi(t) (1 - e^{-t}) + \frac{b(1 - e^{-t})}{t} \right] \\
&= \mathcal{B}_q \left[ \phi(t) (1 - e^{-t}) + \frac{b(1 - e^{-t})}{t} \right] \\
&\quad - \mathcal{P}_{1,q} \mathcal{B}_q \left[ (1 - e^{-t})^{-1} \left( \phi(t) (1 - z e^{-t}) + \frac{b(1 - e^{-t})}{t} \right) \right] \\
&= (1 - \mathcal{P}_{0,q} - \mathcal{P}_{1,q}) \mathcal{B}_q \left[ \frac{b}{t} + \phi \right] = (1 - \mathcal{P}_q^+) f
\end{aligned}$$

where we have used again Proposition 2.5 and Remark 2.7.  $\square$

**Corollary 3.7.** *Let  $z \in \mathbb{C} \setminus (1, \infty)$ . The operator  $\mathcal{Q}_{q,z}$  has an eigenfunction  $g \in \mathcal{H}_q^1$  with eigenvalue  $\lambda_Q = \pm 1$  if and only if  $\mathcal{P}_q^\pm$  has an eigenfunction  $f \in H_{q, \frac{1}{z}}^1$  with eigenvalue  $\lambda_P = \frac{1}{z}$  and term  $c = 0$ . Moreover the eigenfunctions  $g$  and  $f$  satisfy*

$$g = f - z \mathcal{P}_{0,q} f \quad (3.10)$$

*Proof.* If  $f$  is in  $\mathcal{H}_{q,\lambda}^1$  and satisfies  $\mathcal{P}_q^+ f = \lambda f$ , with  $\lambda \neq 1$  then we can apply Theorem 3.6, since by Corollary 2.10 it follows  $b = 0$ . Then by (3.9)  $(1 - \frac{1}{\lambda} \mathcal{P}_{0,q}) f$  is an eigenfunction of  $\mathcal{Q}_{q, \frac{1}{\lambda}}$  with eigenvalue  $\lambda_Q = 1$  if  $c = 0$ . The same follows in the case  $\lambda = 1$ .

On the contrary, if  $g = \mathcal{B}_q[\phi] \in \mathcal{H}_q^1$  satisfies  $\mathcal{Q}_{q,z} g = g$  for  $z \neq 1$ , then the function

$$f := \mathcal{B}_q[(1 - z e^{-t})^{-1} \phi] \in H_q^1$$

satisfies by (3.4)

$$z \mathcal{P}_{1,q} f = g = (1 - z \mathcal{P}_{0,q}) f$$

hence it is an eigenfunction of  $\mathcal{P}_q^+$  with eigenvalue  $\lambda_P = \frac{1}{z}$  and  $c = 0$ . If  $z = 1$  we can repeat the same argument by using the fact that  $\mathcal{Q}_{q,1}$  is defined on  $\tilde{\mathcal{H}}_q^1$ , hence  $g = \mathcal{B}_q[\phi]$  with  $\phi(t) = \phi(0) + O(t^\varepsilon)$  for some  $\varepsilon > 0$  as  $t \rightarrow 0^+$ . Hence we can write  $\frac{\phi(t)}{(1 - e^{-t})} = \frac{b}{t} + \bar{\phi}(t)$ , with  $b = \phi(0)$  and  $\bar{\phi} \in L^1(m_q)$  for all  $\operatorname{Re}(q) > 0$ , and  $f \in H_{q,1}^1$  with  $c = 0$ .  $\square$

**Corollary 3.8.** *Let  $z \in \mathbb{C} \setminus (1, \infty)$ . The eigenfunctions  $g = \mathcal{B}_q[\phi]$  of  $\mathcal{Q}_{q,z}$  with eigenvalue  $\lambda_Q = \pm 1$  are in  $\mathcal{H}_q^2$  and are bounded at  $x = 0$  with  $g(0) = \Gamma(2q) \phi(0)$  if  $z = 1$  and  $g(0) = (1 - z) \Gamma(2q) \phi(0)$  for  $z \neq 1$ .*

*Proof.* Let  $z \neq 1$ . By Corollary 3.7, from any  $g = \mathcal{B}_q[\phi] \in \mathcal{H}_q^1$  eigenfunction of  $\mathcal{Q}_{q,z}$  with eigenvalue  $\lambda_Q = \pm 1$ , we obtain an eigenfunction  $f \in \mathcal{H}_q^1$  of  $\mathcal{P}_q^\pm$  with eigenvalue  $\lambda_P = \frac{1}{z}$  and term  $c = 0$  when written as in (2.27), and (3.10) holds. Moreover, by Corollary 2.10, we know that  $f \in \mathcal{H}_{q, \frac{1}{z}}^2$  and  $b = 0$ , hence actually  $f \in \mathcal{H}_q^2$ . Since  $f = \mathcal{B}_q[(1 - z e^{-t})^{-1} \phi]$  and  $(1 - z e^{-t})^{-1} \in L^\infty$ , then  $\phi \in L^2(m_q)$  and  $g \in \mathcal{H}_q^2$ . If  $z = 1$ , we can repeat the same argument with few corrections. In this case  $f = \mathcal{B}_q[(1 - e^{-t})^{-1} \phi]$  and

$$\frac{\phi(t)}{(1 - e^{-t})} = \frac{\phi(0)}{t} + \bar{\phi}(t) \quad \text{with} \quad \bar{\phi}(t) = \frac{\phi(t) - \phi(0)}{1 - e^{-t}} + \frac{\phi(0)}{1 - e^{-t}} - \frac{\phi(0)}{t}$$



From Corollary 2.10 it follows that  $\bar{\phi}(t) \in L^2(m_q)$ , hence  $\phi \in L^2(m_q)$  and  $g \in \mathcal{H}_q^2$ . Moreover, from (2.6) and (3.10) it follows  $g(x) = zf(x+1)$ , hence  $g(0) = zf(1)$ . Finally from Theorem 2.8, it follows that the eigenfunction  $f$  is written as in (2.20), (2.22), (2.26). Let  $z = 1$ , using (3.10) we write  $g = \mathcal{B}_q[\phi]$  with

$$\phi(t) = e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^n a_n t^n}{\Gamma(n+2q)}$$

with  $\limsup_n \sqrt[n]{|a_n|} \leq 1$ . Since for  $n \geq 0$

$$\mathcal{B}_q[e^{-t}t^n] = \frac{1}{x^{2q}} \int_0^\infty e^{-t(\frac{1}{x}+1)} t^{n+2q-1} dt = \frac{1}{x^{2q}} \mathcal{L}(t^{n+2q-1})|_{\frac{1}{x}+1} = \Gamma(n+2q) \frac{1}{x^{2q}} \left(\frac{x}{x+1}\right)^{n+2q}$$

it follows that

$$g(x) = \frac{1}{(x+1)^{2q}} \sum_{n=0}^{\infty} (-1)^n a_n \left(\frac{x}{x+1}\right)^n \quad (3.11)$$

which implies in particular  $g(0) = \Gamma(2q)\phi(0)$ . The same argument can be used for  $z \neq 1$ .  $\square$

*Remark 3.9.* It follows from Theorem 3.2 that if we restrict the operator  $\mathcal{Q}_{q,1}$  to functions  $g$  which are  $\mathcal{B}_q$  transform of a power series in  $t$ , the residue operator in  $q = \frac{1}{2}$  is given by  $g \mapsto \mathcal{R}_{\frac{1}{2}}g = g(0)$ . This is in accordance with the result in [19].

We finish this section by considering a second induced map from (2.1). It is the *Fibonacci map*  $H$  which is defined by iterating  $F$  once plus the number of times necessary to reach the interval  $[0, 1/2]$ . The map  $H$  is defined by

$$H(x) = \begin{cases} \frac{S_{2n+1}x - S_{2n}}{S_{2n+1} - S_{2n+2}} & \text{if } x \in \left[\frac{S_{2n}}{S_{2n+1}}, \frac{S_{2n+2}}{S_{2n+3}}\right) \\ \frac{S_{2n+1} - S_{2n+2}}{S_{2n+5} - S_{2n+4}} x & \text{if } x \in \left[\frac{S_{2n+3}}{S_{2n+4}}, \frac{S_{2n+1}}{S_{2n+2}}\right] \end{cases} \quad (3.12)$$

where  $\{S_n\}_{n \geq 0}$  are the Fibonacci numbers with  $S_0 = 0$ ,  $S_1 = 1$ . The corresponding operators are obtained by exchanging the roles of  $\mathcal{P}_{0,q}$  and  $\mathcal{P}_{1,q}$  in (3.4). We recall from [5] that the operators  $N_q$  on  $L^2(m_q)$  are of trace class with spectrum

$$\sigma(N_q) = \{0\} \cup \left\{(-1)^k \alpha^{2(q+k)}\right\}_{k \geq 0} \quad (3.13)$$

where  $\alpha = \frac{\sqrt{5}-1}{2}$  and each eigenvalue is simple. Hence we introduce on  $\mathcal{H}_q^1$  the second family of operators

$$\mathcal{R}_{q,z} : \mathcal{H}_q^1 \rightarrow \mathcal{H}(B), \quad z \in \mathbb{C} \setminus \left\{(-1)^k \alpha^{-2(q+k)}\right\}_{k \geq 0} \text{ and } \operatorname{Re}(q) > 0$$

defined by

$$g(x) = \mathcal{B}_q[\phi(t)] \longmapsto (\mathcal{R}_{q,z}g)(x) = z \mathcal{P}_{0,q} \mathcal{B}_q[(1 - z N_q)^{-1} \phi(t)] \quad (3.14)$$

By (3.13) the operators are well defined and we get

**Theorem 3.10.** For  $\operatorname{Re}(q) > 0$ , the operator-valued function  $z \mapsto \mathcal{R}_{q,z}$  is meromorphic in  $\mathbb{C}$  with simple poles at  $\{(-1)^k \alpha^{-2(q+k)}\}_{k \geq 0}$ . Moreover for all  $g \in \mathcal{H}_q^1$  it holds

$$(\mathcal{R}_{q,z}g)(x) = \sum_{n \geq 1} \frac{z^n}{(S_{n+1}x + S_n)^{2q}} g\left(\frac{S_n x + S_{n-1}}{S_{n+1}x + S_n}\right) \quad (3.15)$$

which, by the growth property of the Fibonacci numbers, is absolutely convergent for  $|z| < \alpha^{-2\operatorname{Re}(q)}$ .

*Proof.* The first part follows from the definition of  $\mathcal{R}_{q,z}$  in (3.14) and (3.13). Eq. (3.15) follows instead by writing

$$(\mathcal{R}_{q,z}g)(x) = (z \mathcal{P}_{0,q} (1 - z \mathcal{P}_{1,q})^{-1} g)(x) = \sum_{n \geq 1} z^n (\mathcal{P}_{0,q} \mathcal{P}_{1,q}^{n-1} g)(x)$$

and using

$$(\mathcal{P}_{1,q}^{n-1} g)(x) = \frac{1}{(S_{n-1}x + S_n)^{2q}} g\left(\frac{S_{n-2}x + S_{n-1}}{S_{n-1}x + S_n}\right)$$

which can be proved by induction.  $\square$

*Remark 3.11.* From (3.15) it follows that the operators  $\mathcal{R}_{q,1}$  coincide with the generalised transfer operators of the Fibonacci map  $H$  for functions  $g \in \mathcal{H}(B)$ .

Analogously to  $\mathcal{Q}_{q,z}$  we now study the relations between  $\mathcal{R}_{q,z}$  and  $\mathcal{P}_q^\pm$ .

**Theorem 3.12.** Let  $f \in \mathcal{H}_{q,\mu}^2$ , with  $c = b = 0$ . Then for  $z = \frac{1}{\mu} \in \mathbb{C} \setminus \{(-1)^k \alpha^{-2(q+k)}\}$

$$(1 \mp \mathcal{R}_{q,z}) (1 - z \mathcal{P}_{1,q}) f = (1 - z \mathcal{P}_q^\pm) f \quad (3.16)$$

*Proof.* First of all, that the left-hand side of (3.16) is well defined follows easily from definitions. Notice that in this case we need  $c = b = 0$  to be sure that  $(1 - z \mathcal{P}_{1,q}) f$  is in  $\mathcal{H}_q^1$ . Hence let  $f = \mathcal{B}_q[\phi]$  with  $\phi \in L^2(m_q)$ . Applying  $(1 - \mathcal{R}_{q,z})$ . It follows

$$\begin{aligned} (1 \mp \mathcal{R}_{q,z}) (1 - z \mathcal{P}_{1,q}) f &= (1 \mp \mathcal{R}_{q,z}) \mathcal{B}_q[(1 - z N_q)\phi(t)] = \\ &= \mathcal{B}_q[(1 - z N_q - z M)\phi(t)] = (1 - z \mathcal{P}_q^\pm) \mathcal{B}_q[\phi]. \end{aligned}$$

$\square$

We remark that using the power series expansion (3.15), one can prove that relation (3.16) holds for all  $f \in \mathcal{H}(B)$  for  $|z| < \alpha^{-2\operatorname{Re}(q)}$ .

**Corollary 3.13.** Let  $z \in \mathbb{C} \setminus \{(-1)^k \alpha^{-2(q+k)}\}$ . The operator  $\mathcal{R}_{q,z}$  has an eigenfunction  $g \in \mathcal{H}_q^2$  with eigenvalue  $\lambda_R = \pm 1$  if and only if  $\mathcal{P}_q^\pm$  has an eigenfunction  $f$  with eigenvalue  $\lambda_P = \frac{1}{z}$  and  $f \in \mathcal{H}_{q,\mu}$  with  $c = b = 0$ .

*Proof.* The proof is as in Corollary 3.7.  $\square$

Putting together Corollaries 3.7 and 3.13, it follows that

**Corollary 3.14.** *Let  $z \in \mathbb{C} \setminus ((1, \infty) \cup \{(-1)^k \alpha^{-2(q+k)}\})$ . Then  $f \in \mathcal{H}_{q,\mu}^2$  with  $c = b = 0$  is an eigenfunction of  $\mathcal{P}_q^\pm$  with eigenvalue  $\lambda_P = \frac{1}{z} = \mu$  if and only if*

$$f(x) = h_0(x) + h_1(x)$$

with  $h_0$  and  $h_1$  in  $\mathcal{H}_q^2$  and eigenfunctions of  $\mathcal{Q}_{q,z}$  and  $\mathcal{R}_{q,z}$  respectively, with eigenvalues  $\lambda_Q = \lambda_R = \pm 1$ .

*Proof.* Simply use that  $h_0 := (1 - z\mathcal{P}_{0,q})f$  and  $h_1 := (1 - z\mathcal{P}_{1,q})f$ .  $\square$

*Remark 3.15.* Recall that relation (3.16) can be extended to functions in  $\mathcal{H}(B)$ , and that Corollary 3.7 holds for functions  $f \in \mathcal{H}_{q,1}^2$  with  $b$  not necessarily null. Then an example of functions  $f, h_0, h_1$  is given for  $q = 1$  by

$$\frac{1}{x} = \frac{1}{1+x} + \frac{1}{x(x+1)}$$

## 4 Two-variable zeta functions of Ruelle and Selberg

The first two-variable zeta function we are interested in can be written in terms of the Gauss map as

$$Z(q, z) := \exp \left( - \sum_{n \geq 1} \frac{z^n}{n} \sum_{x=G^{2n}(x)} \frac{|(G^{2n})'(x)|^{-q}}{1 - |(G^{2n})'(x)|^{-1}} \right) \quad (4.1)$$

For  $z = 1$  and  $\text{Re}(q) > 1/2$  the function  $Z(q, 1)$  coincides with the *Selberg zeta function* for the full modular group. This follows from the well known one-to-one correspondence between the length spectrum (with multiplicities) of the modular surface  $PSL(2, \mathbb{Z}) \backslash \mathbb{H}$  and the set of values  $\log |(G^{2n})'(x)|$  (see e.g. [25]). Another zeta function which naturally comes into the play [24] is the *Ruelle zeta function* of the Farey map  $F$ , defined for  $|z|$  small enough by

$$\zeta(q, z) := \exp \left( \sum_{n \geq 1} \frac{z^n}{n} \sum_{x=F^n(x)} |(F^n)'(x)|^{-q} \right) \quad (4.2)$$

We shall study these functions by means of the operator-valued functions dealt with in the previous section. Our approach is similar in spirit to that used in [20] for  $Z(q, 1)$ . But first we describe the correspondence between the periodic points of the map  $F$  and those of its induced versions  $G$  (3.1) and  $H$  (3.12). Denoting  $\text{Per } F$ ,  $\text{Per } G$  and  $\text{Per } H$  the corresponding subsets of  $[0, 1]$  we have

$$\text{Per } F \setminus \{0\} \cup \{\alpha\} = \text{Per } G \setminus \{\alpha\} = \text{Per } H \setminus \{0\}$$

From the definitions we immediately see that whenever  $x$  belongs to either of these sets its continued fraction expansion has to be periodic, which we write

$$x = [\overline{a_1, \dots, a_n}]$$

Denoting  $p_F(x)$ ,  $p_G(x)$  and  $p_H(x)$  the corresponding periods we have

$$p_F(x) = \sum_{i=1}^n a_i \quad , \quad p_G(x) = n \quad , \quad p_H(x) = p_F(x) - \#\{i \in [1, n] : a_i = 1\}$$

In other words, if for a given map  $T : [0, 1] \rightarrow [0, 1]$  we define the *partition function*

$$Z_n(q, T) := \sum_{x=T^n(x)} |(T^n)'(x)|^{-q}$$

then

$$Z_n(q, F) = 1 + \sum_{m=1}^n \frac{n}{m} Z_m(q, G) = \alpha^{2qn} + \sum_{m=1}^n \frac{n}{m} Z_m(q, H)$$

Let moreover

$$\lambda(q) := \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(q, F) \quad (4.3)$$

It follows from thermodynamic formalism that the above limit exists for all  $q \in \mathbb{R}$  and is a differentiable and monotonically decreasing function for  $q \in (-\infty, 1)$ , with  $\lim_{q \rightarrow 1^-} \lambda(q) = 1$  and  $\lambda(q) = 1$  for all  $q \geq 1$  ([22]). In particular, for  $q \in (-\infty, 1)$  the function  $\zeta(q, z)$  converges absolutely for  $|z| < 1/\lambda(q)$  and has a simple pole at  $z = 1/\lambda(q)$ .

Our aim is now to express both  $Z(q, z)$  and  $\zeta(q, z)$  in terms of Fredholm determinants of the operators  $\mathcal{Q}_{q,z}$  introduced in Section 3. Following Mayer [18], we restrict the operators  $\mathcal{Q}_{q,z}$  to the Banach space  $A_\infty(D)$  of functions which are holomorphic on  $D$  and continuous on  $\overline{D}$  where

$$D := \left\{ x \in \mathbb{C} : |x - 1| < \frac{3}{2} - \varepsilon \right\}.$$

for small  $\varepsilon > 0$ . The space  $A_\infty(D)$  is the set of holomorphic functions on which it is natural to study the spectral properties of  $\mathcal{Q}_{q,z}$  written as in Theorem 3.4. We now prove that

**Proposition 4.1.** *If  $g$  is in  $A_\infty(D)$  then  $g$  is in  $\tilde{\mathcal{H}}_q^2$  for all  $\text{Re}(q) > 0$ , and moreover  $g = \mathcal{B}_q[\phi]$  with  $\phi(t) = \phi(0) + O(t)$  as  $t \rightarrow 0^+$ .*

In particular since  $A_\infty(D) \subset \tilde{\mathcal{H}}_q^2$  we can study the action of  $\mathcal{Q}_{q,z}$  on  $A_\infty(D)$  for all  $z \in \mathbb{C} \setminus (1, +\infty)$  and  $\text{Re}(q) > 0$  by means of the properties stated in Section 3.

*Proof of Proposition 4.1.* Let us first assume  $q$  real and positive. We use the characterization of the space  $L^2(m_q)$  introduced in [5]. The Hilbert space  $L^2(m_q)$  admits as orthogonal basis the set  $\{e_n^q(t)\}$  of the generalized Laguerre polynomials

$$e_n^q(t) := L_n^{2q-1}(t) = \sum_{m=0}^n \frac{\Gamma(n+2q)}{\Gamma(m+2q)(n-m)!} \frac{(-t)^m}{m!}$$

which satisfy

$$\begin{aligned} \int_0^\infty e_n^q(t) e_m^q(t) t^{2q-1} e^{-t} dt &= \frac{\Gamma(n+2q)}{n!} \delta_{n,m} \\ \mathcal{B}_q[e_n^q(t)] &= \frac{\Gamma(n+2q)}{n!} (-1)^n (x-1)^n \end{aligned} \quad (4.4)$$

Let now  $g(x)$  be in  $A_\infty(D)$ . Then it can be expressed as a power series

$$g(x) = \sum_{n=0}^\infty a_n (x-1)^n \quad a_n \in \mathbb{C}$$

with

$$\limsup \sqrt[n]{|a_n|} \leq \left(\frac{3}{2} - \varepsilon\right)^{-1}.$$

Hence from (4.4) we can write

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \mathcal{B}_q[e_n^q]$$

Now let

$$\phi(t) := \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} e_n^q(t)$$

which is in  $L^2(m_q)$  since

$$\int_0^{\infty} |\phi(t)|^2 t^{2q-1} e^{-t} dt = \sum_{n=0}^{\infty} \frac{|a_n n!|^2}{|\Gamma(n+2q)|^2} \frac{\Gamma(n+2q)}{n!} < \infty$$

and satisfies

$$\begin{aligned} \phi(0) &= \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} e_n^q(0) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \frac{\Gamma(n+2q)}{\Gamma(2q) n!} < \infty \\ \lim_{t \rightarrow 0^+} \frac{\phi(t) - \phi(0)}{t} &= - \sum_{n=0}^{\infty} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \frac{\Gamma(n+2q)}{\Gamma(2q+1)(n-1)!} < \infty \end{aligned}$$

To finish the proof we have to show that  $g = \mathcal{B}_q[\phi]$ . This follows from

$$\begin{aligned} \left| \mathcal{B}_q[\phi] - \sum_{n=0}^N \frac{(-1)^n a_n n!}{\Gamma(n+2q)} \mathcal{B}_q[e_n^q] \right| &\leq \int_0^{\infty} \left| e^{-\frac{t}{x}} t^{2q-1} \sum_{n>N} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} e_n^q(t) \right| dt \leq \\ &\leq \int_0^{\infty} \left| e^{-t} t^{2q-1} \sum_{n>N} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} e_n^q(t) \right| dt = \int_0^{\infty} \left| \sum_{n>N} \frac{(-1)^n a_n n!}{\Gamma(n+2q)} e_n^q(t) \right| m_q(dt) \end{aligned}$$

and the last term vanishes as  $N \rightarrow \infty$  since  $\phi \in L^2(m_q) \subset L^1(m_q)$ .

If  $q$  is not real then let  $q = \xi + \eta i$  with  $\xi > 0$ , and repeat the same argument as above in the space  $L^2(m_\xi)$ . The obvious inclusion  $L^2(m_\xi) \subset L^2(m_q)$  finishes the proof.  $\square$

We now recall that by (3.11) in the proof of Corollary 3.8, the eigenfunctions  $g \in \tilde{\mathcal{H}}_q^2$  of  $\mathcal{Q}_{q,z}$  are in  $A_\infty(D)$ . Hence putting together Remark 3.1, Theorem 3.2 and Proposition 4.1, we get

**Theorem 4.2.** *For  $z \in \mathbb{C} \setminus [1, \infty)$  and  $\operatorname{Re}(q) > 0$ , the operators  $\mathcal{Q}_{q,z}$  on  $A_\infty(D)$  are isomorphic to the operators  $Q_{q,z}$  defined in (3.5). In particular for  $\operatorname{Re}(q) > 0$  the operator-valued function  $z \mapsto Q_{q,z}$  is analytic in  $\mathbb{C} \setminus [1, +\infty)$  and for  $z \in \mathbb{C} \setminus [1, +\infty)$ , the operator-valued function  $q \mapsto Q_{q,z}$  is analytic in  $\operatorname{Re}(q) > 0$ .*

*For  $z = 1$  and  $\operatorname{Re}(q) > \frac{1}{2}$ , the operators  $\mathcal{Q}_{q,1}$  on  $A_\infty(D)$  are isomorphic to the operators  $Q_{q,1}$  defined in (3.5). Moreover the operator-valued function  $q \mapsto Q_{q,1}$  can be extended to a meromorphic function in  $\operatorname{Re}(q) > 0$  via the formula*

$$Q_{q,1}\phi = N_q(1-M)^{-1}\phi(0) + N_q(1-M)^{-1}(\phi(t) - \phi(0)) \quad (4.5)$$

*which has a simple pole at  $q = \frac{1}{2}$  with residue the rank 1 operator  $\phi \mapsto (R_{\frac{1}{2}}\phi)(x) = \phi(0)$ .*

For the existence of the first term in (4.5), we have used (2.15). We now use the following

**Theorem 4.3** ([5]). *For  $\operatorname{Re}(q) > 0$ , the operators  $\mathcal{P}_{1,q}$  and  $N_q$  on the spaces  $\mathcal{H}_q^2$  and  $L^2(m_q)$ , respectively, are of trace class.*

From this and Theorem 4.2 we immediately obtain

**Corollary 4.4.** *The operators  $\mathcal{Q}_{q,z}$  on  $A_\infty(D)$  are of trace class and for  $z \in \mathbb{C} \setminus [1, \infty)$  and  $\operatorname{Re}(q) > 0$ , and for  $\operatorname{Re}(q) > \frac{1}{2}$  if  $z = 1$  it holds*

$$\operatorname{trace}(\mathcal{Q}_{q,z}) = \operatorname{trace}(Q_{q,z}) = z \int_0^\infty \frac{J_{2q-1}(2t)}{t^{2q-1}} (1 - ze^{-t})^{-1} m_q(dt) \quad (4.6)$$

using (2.13).

Applying Fredholm theory [12] to the operators  $\mathcal{Q}_{q,z}$  we conclude that

**Corollary 4.5.** *For  $z \in \mathbb{C} \setminus [1, \infty)$  the Fredholm determinants  $q \mapsto \det(1 \pm \mathcal{Q}_{q,z})$  are analytic functions in  $\operatorname{Re}(q) > 0$ . For  $\operatorname{Re}(q) > 0$  the the Fredholm determinants  $z \mapsto \det(1 \pm \mathcal{Q}_{q,z})$  are analytic functions in  $z \in \mathbb{C} \setminus [1, \infty)$ . For  $z = 1$  the determinants  $q \mapsto \det(1 \pm \mathcal{Q}_{q,1})$  are analytic functions in  $\operatorname{Re}(q) > \frac{1}{2}$  with a meromorphic extension to  $\operatorname{Re}(q) > 0$  with a simple pole at  $q = \frac{1}{2}$ .*

Using Corollary 4.5 we can express  $Z(q, z)$  and  $\zeta(q, z)$  in terms of the Fredholm determinants  $\det(1 \pm \mathcal{Q}_{q,z})$ . By (4.6), this is an easy generalisation of results from [13, Section 4] and [18, 20]. More precisely we have

**Theorem 4.6.** *For  $z \in \mathbb{C} \setminus [1, \infty)$  and  $\operatorname{Re}(q) > 0$  it holds*

$$Z(q, z) = \det[(1 - \mathcal{Q}_{q,z})(1 + \mathcal{Q}_{q,z})] \quad (4.7)$$

and

$$\zeta(q, z) = (1 - z)^{-1} \frac{\det(1 + \mathcal{Q}_{q+1,z})}{\det(1 - \mathcal{Q}_{q,z})} \quad (4.8)$$

as analytic, respectively meromorphic, functions. Moreover

$$(q, z) \mapsto \zeta(q, z) Z(q, z) = (1 - z)^{-1} \det(1 + \mathcal{Q}_{q+1,z}) \det(1 + \mathcal{Q}_{q,z})$$

is analytic in  $\{z \in \mathbb{C} \setminus [1, \infty)\} \times \{q \in \mathbb{C} : \operatorname{Re}(q) > 0\}$ .

For  $z = 1$  the function  $Z(q, 1)$  satisfies (4.7) and is analytic for  $\operatorname{Re}(q) > \frac{1}{2}$ . Moreover it can be continued to  $\operatorname{Re}(q) > 0$  as a meromorphic function with a simple pole at  $q = \frac{1}{2}$ .

The case  $z = 1$  for the Ruelle zeta function  $\zeta(q, z)$  is more delicate as is evident from the term  $(1 - z)^{-1}$  in (4.8). However at the end of Section 5 we obtain that it is possible to define  $\zeta(q, 1)$  for  $\operatorname{Re}(q) > 1$ .

**Remark 4.7.** For  $q$  real there will be a simple pole, respectively a simple zero, at  $z = \frac{1}{\lambda(q)} \in (\frac{1}{2}, 1)$  with  $\lambda(q)$  defined in (4.3). More information on poles, respectively zeroes, for the zeta functions can be obtained by the study of the point spectrum of the operators  $\mathcal{P}_q^\pm$  in  $L^2(m_q)$ . Indeed by Corollary 3.7, eigenfunctions of  $\mathcal{P}_q^+$  in  $L^2(m_q)$  with eigenvalue  $\frac{1}{z} \notin (0, 1]$  are in one-to-one correspondence with eigenfunctions of  $\mathcal{Q}_{q,z}$  in  $\tilde{\mathcal{H}}_q^2$  with eigenvalue 1. For real  $q$ , the spectrum of  $\mathcal{P}_q^+$  in  $L^2(m_q)$  has been studied numerically in [21]. The results suggest that  $\lambda(q) \in (1, 2)$  for  $q \in (0, 1)$  is the only

eigenvalue, with the rest of the spectrum given by the interval  $[0, 1]$  which is purely continuous (see also [5]). Whereas there are no eigenvalues for  $q \geq 1$  and the spectrum is purely continuous. This would imply that for  $q \in (0, 1)$  the function  $z \mapsto \zeta(q, z)$  has only one simple pole at  $z = \frac{1}{\lambda(q)}$  and has no poles for  $q \geq 1$ , and respectively the function  $z \mapsto Z(q, z)$  has only one single zero at  $z = \frac{1}{\lambda(q)}$  for  $q \in (0, 1)$  and no zeroes for  $q \geq 1$ .

#### 4.1 Remarks on the case $z = 1$

We now give some remarks on the relations with the works of Mayer and Lewis-Zagier, who have studied the case  $z = 1$ . We first mention that in [20] the meromorphic extension for  $Z(q, 1)$  that we have obtained in Theorem 4.6 is given to all the complex  $q$ -plane. Moreover, (4.7) and (4.8) give an explicit connection between zeroes of  $Z(q, z)$ , respectively zeroes or poles of  $\zeta(q, z)$ , and the existence of eigenfunctions  $g \in A_\infty(D)$  for  $\mathcal{Q}_{q,z}$  with eigenvalue  $\lambda_Q = \pm 1$ . By Corollary 3.7 this turns out to be a connection with eigenfunctions of  $\mathcal{P}_q^\pm$  with eigenvalues  $\lambda_P = \frac{1}{z}$ . These connections have been proved in [20] for the operators  $\mathcal{Q}_{q,1}$ , and in [17] for the operators  $\mathcal{P}_q^\pm$  with  $\lambda_P = 1$  (see also Proposition 2.1).

Finally the characterisation of the eigenfunctions for  $\mathcal{P}_q^\pm$  given in Corollary 2.10 together with results from [17] lead to a new proof of the following result from [9]

**Theorem 4.8.** *Even, respectively odd, spectral zeroes of  $Z(q, 1)$  correspond to eigenfunctions of  $\mathcal{P}^+$ , respectively  $\mathcal{P}^-$ . Moreover the zeroes  $q$  in  $\text{Re}(q) > 0$  such that for the Riemann zeta function it holds  $\zeta_R(2q) = 0$ , correspond to eigenfunctions of  $\mathcal{P}_q^+$ , hence of  $\mathcal{Q}_{q,1}$  with eigenvalue  $\lambda_Q = 1$ .*

*Proof.* By Corollary 3.7 and Theorem 4.6, the zeroes of  $Z(q, 1)$  correspond to eigenfunctions of  $\mathcal{P}_q^\pm$  with eigenvalue  $\lambda_P = 1$  and term  $c = 0$  in (2.27). From Corollary 2.10 it also follows that for eigenfunctions of  $\mathcal{P}_q^-$  it holds  $b = 0$ . Hence Corollary 2.10 and (2.6) imply that

$$\mathcal{P}_q^- f = f \text{ with } c = 0 \Rightarrow f(x) = \begin{cases} O(1) & x \rightarrow 0 \\ O(x^{-2\text{Re}(q)}) & x \rightarrow \infty \end{cases}$$

For eigenfunctions of  $\mathcal{P}_q^+$  instead two cases are possible,  $b = 0$  and  $b \neq 0$ . We have

$$\begin{aligned} \mathcal{P}_q^+ f = f \text{ with } c = b = 0 &\Rightarrow f(x) = \begin{cases} O(1) & x \rightarrow 0 \\ O(x^{-2\text{Re}(q)}) & x \rightarrow \infty \end{cases} \\ \mathcal{P}_q^+ f = f \text{ with } c = 0, b \neq 0 &\Rightarrow f(x) = \begin{cases} \sim \frac{b}{2q-1} \frac{1}{x} & x \rightarrow 0 \\ \sim \frac{b}{2q-1} x^{1-2\text{Re}(q)} & x \rightarrow \infty \end{cases} \end{aligned}$$

Applying Theorem 2 and the subsequent Corollary from [17] it follows that all eigenfunctions of  $\mathcal{P}_q^-$  with  $c = 0$  are period functions associated to cusp forms or Maass wave forms. Moreover from Proposition 2.1-(ii) it follows that they are odd period functions, in the sense of [17], hence are associated to odd cusp forms. The same holds for all eigenfunction of  $\mathcal{P}_q^+$  with  $c = b = 0$  which are associated to even cusp forms. This concludes the proof for the spectral zeroes of  $Z(q, 1)$ . The previous argument also implies that the other zeroes of the Selberg zeta function with  $\text{Re}(q) > 0$  necessarily correspond to eigenfunctions of  $\mathcal{P}_q^+$  with  $c = 0$  and  $b \neq 0$ . For example it is well known that the zero at  $q = 1$  corresponds to the eigenfunction  $f(x) = \frac{1}{x}$  of  $\mathcal{P}_1^+$  and the zeroes satisfying  $\zeta_R(2q) = 0$  correspond to the eigenfunctions  $f_q^+(x)$  defined in (2.34).  $\square$

## 5 Connections with Farey fractions

We now use (4.7) and (4.8) to give expressions for the generalised Selberg and Ruelle zeta functions as exponentials of Dirichlet series. By (4.7), (4.8) and Theorem 3.6 we can write for  $z \in \mathbb{C} \setminus [1, \infty)$  and  $\operatorname{Re}(q) > 0$

$$Z(q, z) = \det \left[ 1 - z \mathcal{P}_{1,q} (1 - z \mathcal{P}_{0,q})^{-1} \right] \det \left[ 1 + z \mathcal{P}_{1,q} (1 - z \mathcal{P}_{0,q})^{-1} \right] \quad (5.1)$$

$$\zeta(q, z) = (1 - z)^{-1} \frac{\det \left[ 1 + z \mathcal{P}_{1,q+1} (1 - z \mathcal{P}_{0,q+1})^{-1} \right]}{\det \left[ 1 - z \mathcal{P}_{1,q} (1 - z \mathcal{P}_{0,q})^{-1} \right]} \quad (5.2)$$

The existence of the right-hand sides can be justified as in [23]. Expression (5.1) can be given also for  $Z(q, 1)$  and it follows from the uniform convergence of the series representation of  $\mathcal{Q}_{q,z}$  on  $A_\infty(D)$  given in Theorem 3.4. We now perform a formal calculation which gives a connection between the zeta functions  $\zeta(q, z)$  and  $Z(q, z)$  and the Farey fractions.

Let us consider the matrices

$$\phi_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \phi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

as elements of the group  $GL(2, \mathbb{Z})$  acting on  $\mathbb{C}$  as Möbius transformations, see [2]. Then  $(\mathcal{P}_{i,q} f)(x) = (\phi'_i(x))^q f(\phi_i(x))$ . Thus, each term in the expansion of  $(\mathcal{P}_q^\pm)^n$  can be represented in terms of a product of the matrices  $\phi_i$ . At the same time, the Farey fractions can be represented by means of a subgroup of  $SL(2, \mathbb{Z})$  with generators

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

**Lemma 5.1.** *Expanding*

$$(\mathcal{P}_q^+)^n + (\mathcal{P}_q^-)^n - 2(\mathcal{P}_{0,q})^n$$

one obtains  $2(2^{n-1} - 1)$  terms which are twice all the possible combinations of  $n$  factors involving  $L$  and  $R$  and starting with  $L$ , without the term  $L^n$ .

*Proof.* The only products which do not cancel out are those where  $\mathcal{P}_{1,q}$  appears an even numbers of times (counted twice). On the other hand we point out that if we denote  $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  then  $K^2 = Id$ ,  $L = \phi_0$  and  $LK = KR = \phi_1$ . We thus have

$$\phi_1 \underbrace{\phi_0 \dots \phi_0}_n \phi_1 = LK \underbrace{L \dots L}_n KR = L \underbrace{R \dots R}_n R$$

The thesis now easily follows. □

**Proposition 5.2.** *Let  $A$  be the matrix corresponding to the term*

$$(\mathcal{P}_{0,q})^{n_1} (\mathcal{P}_{1,q})^{n_2} \dots (\mathcal{P}_{i,q})^{n_l}$$

with  $\sum_{j=1}^l n_j = n > 1$  and  $i = (1 + (-1)^l)/2$ . Then, setting  $T := \operatorname{trace}(A)$ , we have  $T > 2$  and

$$\operatorname{trace} [(\mathcal{P}_{0,q})^{n_1} (\mathcal{P}_{1,q})^{n_2} \dots (\mathcal{P}_{i,q})^{n_l}] = \frac{1}{\sqrt{T^2 - 4}} \left( \frac{2}{T + \sqrt{T^2 - 4}} \right)^{2q-1}$$



*Proof.* The proof of the first assertion amounts to a straightforward verification. Let  $\mathcal{V}$  be the composition operator acting as  $(\mathcal{V}f)(x) = \varphi(x)f(\psi(x))$ . If  $\psi(x)$  is holomorphic in a disk and has there a unique fixed point  $\bar{x}$  with  $|\psi'(\bar{x})| < 1$  then  $\mathcal{V}$  is of the trace-class with  $\text{trace}(\mathcal{V}) = \frac{\varphi(\bar{x})}{1-\psi'(\bar{x})}$ . The thesis follows by applying this relation to the operators under consideration.  $\square$

We are now going to make use of the correspondence between products of matrices  $L$  and  $R$  and the Farey fractions. Let us consider the Farey tree  $\mathcal{F}$ . We recall that every rational number  $\frac{a}{b} \in (0, 1)$  appears exactly once in  $\mathcal{F}$ . One may therefore identify  $\frac{a}{b}$  with the path on  $\mathcal{F}$  which reaches it starting from the root node  $\frac{1}{2}$  (first row), which in turn can be encoded as a matrix product in the following way: first recall that every rational number  $\frac{a}{b} \in (0, 1)$  has a unique finite continued fraction expansion  $\frac{a}{b} = [a_1, \dots, a_k]$  with  $a_k > 1$ , and one may define the rank of  $\frac{a}{b}$  as

$$\frac{a}{b} = [a_1, \dots, a_k] \implies \text{rank} \left( \frac{a}{b} \right) = \sum_{i=1}^k a_i - 1$$

It turns out that  $\frac{a}{b}$  has rank  $n$  if and only if it belongs to  $\mathcal{F}_{n+1} \setminus \mathcal{F}_n$ . Furthermore, we may uniquely decompose  $\frac{a}{b}$  as

$$\frac{a}{b} = \frac{a' + a''}{b' + b''} \quad \text{with} \quad b'a - a'b = ba'' - ab'' = b'a'' - a'b'' = 1$$

The neighbours  $\frac{a'}{b'}$  and  $\frac{a''}{b''}$  are the *parents* of  $\frac{a}{b}$  in  $\mathcal{F}$  and we may accordingly identify

$$\frac{a}{b} \simeq \begin{pmatrix} a'' & a' \\ b'' & b' \end{pmatrix} \in \mathcal{Z}$$

where

$$\mathcal{Z} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : 0 < a \leq c, 0 \leq b < d \right\}$$

Clearly  $\frac{1}{2} \simeq L$  and, more generally,

$$\frac{a}{b} \simeq L \prod_i M_i \tag{5.3}$$

where the number of terms in the product  $L \prod_i M_i$  is equal to  $\text{rank}(\frac{a}{b})$  and  $M_i = L$  or  $M_i = R$  according to whether the  $i$ -th turn, along the descending path in  $\mathcal{F}$  which starts from the root node  $\frac{1}{2}$  and reaches  $\frac{a}{b}$ , goes to the left or to the right. Using (5.3) one may then define a map  $T : \mathcal{F} \rightarrow \mathbb{N}$  as

$$T \left( \frac{a}{b} \right) := \text{trace} \left( L \prod_i M_i \right) \tag{5.4}$$

Note moreover that the set

$$\tilde{\mathcal{F}}_n := \mathcal{F}_{n+1} \setminus \mathcal{F}_n = \left\{ \frac{a}{b} \in \mathcal{F} : \text{rank} \left( \frac{a}{b} \right) = n \right\}$$

has  $2^{n-1}$  elements which are in a one-to-one correspondence with the (equal pairs of) elements in the expansion dealt with in Lemma 5.1 plus  $\left\{ \frac{1}{n+1} \right\}$ . We now obtain an expression for the  $Z(q, z)$  and  $\zeta(q, z)$  as exponentials of power series whose coefficients are computed along lines of  $\mathcal{F}$ .

**Theorem 5.3.** For  $\operatorname{Re}(q) > 1$  and  $|z| \leq 1$  the two-variable zeta functions  $Z(q, z)$  and  $\zeta(q, z)$  can be written as

$$Z(q, z) = \exp \left( - \sum_{n \geq 2} \frac{z^n}{n} \Lambda_n(q) \right) \quad , \quad \zeta(q, z) = \exp \left( \sum_{n \geq 1} \frac{z^n}{n} \Xi_n(q) \right)$$

with

$$\begin{aligned} \Lambda_n(q) &:= \sum_{\frac{a}{b} \in \tilde{\mathcal{F}}_n \setminus \left\{ \frac{1}{n+1} \right\}} \frac{2}{\sqrt{T^2(\frac{a}{b}) - 4}} \left( \frac{2}{T(\frac{a}{b}) + \sqrt{T^2(\frac{a}{b}) - 4}} \right)^{2q-1} \\ \Xi_n(q) &= \sum_{\frac{a}{b} \in \tilde{\mathcal{F}}_n} \left[ \left( \frac{2}{T(\frac{a}{b}) + \sqrt{T^2(\frac{a}{b}) - 4}} \right)^{2q} + \left( \frac{2}{T(\frac{a}{b}) + \sqrt{T^2(\frac{a}{b}) + 4}} \right)^{2q} \right] \end{aligned}$$

where the map  $T$  is defined in (5.3)-(5.4).

*Proof.* We first obtain the expansion for  $Z(q, z)$ . A formal manipulation of (5.1) gives

$$Z(q, z) = \det[(1 - z\mathcal{P}_q^+)(1 - z\mathcal{P}_q^-)(1 - z\mathcal{P}_{0,q})^{-2}] \quad (5.5)$$

which is well defined by Lemma 5.1.

We now only have to prove that  $\sum_{n \geq 2} \frac{z^n}{n} \Lambda_n(q)$  converge for  $\operatorname{Re}(q) > 1$  and  $|z| \leq 1$ . Since then the assertion follows from (5.5), Proposition 5.2 and the definition of the map  $T$  in (5.3)-(5.4). Set

$$\gamma(k, n) := \# \left\{ \frac{a}{b} \in \tilde{\mathcal{F}}_n \setminus \left\{ \frac{1}{n+1} \right\} : T\left(\frac{a}{b}\right) = k \right\}$$

and let  $\mathcal{M}$  be the free multiplicative monoid generated by the matrices  $L$  and  $R$ . The function

$$\Psi(k) = \#\{X \in \mathcal{M} : \operatorname{trace}(X) \leq k\}$$

has been recently studied in the literature and the asymptotic behaviour

$$\Psi(k) = \frac{k^2 \log k}{\zeta_R(2)} + O(k^2)$$

has been found (see [4, 14]). Let us decompose  $\Psi(k)$  as  $\Psi(k) = \Psi_L(k) + \Psi_R(k)$  where  $\Psi_L(k)$  (resp.  $\Psi_R(k)$ ) is obtained restricting to the elements of  $\mathcal{M}$  which start with  $L$  (respectively  $R$ ). Note that

$$\Psi_L(k) = \sum_{j \leq k} \sum_{n=2}^{j-1} \gamma(j, n)$$

Moreover, using  $R^k K = K L^k$  one easily realises that if

$$\frac{a}{b} \simeq L \prod_i M_i = \begin{pmatrix} a'' & a' \\ b'' & b' \end{pmatrix}$$

then, setting  $\overline{M}_i = L$  if  $M_i = R$  and viceversa, we have

$$\frac{b}{a} \simeq R \prod_i \overline{M}_i = \begin{pmatrix} b' & b'' \\ a' & a'' \end{pmatrix}$$

Therefore  $T\left(\frac{a}{b}\right) = T\left(\frac{b}{a}\right)$  and

$$\sum_{j \leq k} \sum_{n=2}^{j-1} \gamma(j, n) = \frac{k^2 \log k}{2\zeta_R(2)} + O(k^2)$$

This implies that if

$$\alpha(k) := \sum_{n=2}^{k-1} \frac{\gamma(k, n)}{n}$$

then

$$\sum_{j \leq k} \alpha(j) = O(k^2 \log k)$$

hence

$$\sum_{n \geq 2} \frac{1}{n} \Lambda_n(q) = \sum_{k=3}^{\infty} \frac{2\alpha(k)}{\sqrt{k^2 - 4}} \left( \frac{2}{k + \sqrt{k^2 - 4}} \right)^{2q-1}$$

converges absolutely for  $\operatorname{Re}(q) > 1$ . The case  $|z| < 1$  easily follows.

To obtain the expansion for  $\zeta(q, z)$ , we again start from the following formal manipulation of (5.2)

$$\zeta(q, z) = \frac{\det[(1 - z\mathcal{P}_{q+1}^-)(1 - z\mathcal{P}_q^+)^{-1}(1 - z\mathcal{P}_{0,q+1})^{-1}(1 - z\mathcal{P}_{0,q})]}{(1 - z)} \quad (5.6)$$

Now, given  $\epsilon > 0$  let us consider the perturbation  $M_{q,\epsilon}$  of the operator  $M$  in (2.12) acting as

$$(M_{q,\epsilon} \varphi)(t) = \frac{e^{-(\frac{1-\epsilon}{1+\epsilon})t}}{(1+\epsilon)^q} \varphi\left(\frac{t}{1+\epsilon}\right)$$

Reasoning as in the proof of [11, Proposition 4.5] one easily sees that for all  $\epsilon > 0$  the operator  $M_{q,\epsilon}$  is of trace class on  $L^2(m_q)$  and its spectrum is given by  $\sigma(M_{q,\epsilon}) = \{(1+\epsilon)^{-q-k}\}_{k \geq 0}$ , each eigenvalue being simple. Therefore

$$\operatorname{trace}(M_{q,\epsilon}^n - M_{q+1,\epsilon}^n) = (1+\epsilon)^{-q-n}$$

Let moreover  $\mathcal{P}_{0,q}^\epsilon$  be the corresponding operator on  $\mathcal{H}_q^2$ . A short calculation then gives

$$\lim_{\epsilon \rightarrow 0^+} \det[(1 - z\mathcal{P}_{0,q+1}^\epsilon)^{-1}(1 - z\mathcal{P}_{0,q}^\epsilon)] = \lim_{\epsilon \rightarrow 0^+} \left(1 - \frac{z}{1+\epsilon}\right)^{(1+\epsilon)^{-q}} = 1 - z$$

and therefore, letting  $\mathcal{P}_{q,\epsilon}^\pm := \mathcal{P}_{0,q}^\epsilon \pm \mathcal{P}_{1,q}$ ,

$$\zeta(q, z) = \lim_{\epsilon \rightarrow 0^+} \det[(1 - z\mathcal{P}_{q+1,\epsilon}^-)(1 - z\mathcal{P}_{q,\epsilon}^+)^{-1}] \quad (5.7)$$

Starting from this expression along with [8, Corollary 3.11] and proceeding as above one readily obtains the expansion for  $\zeta(q, z)$ .  $\square$

We remark that from Theorem 5.3, in particular from (5.7), we obtain a definition for  $\zeta(q, 1)$  for  $\operatorname{Re}(q) > 1$ , which is not immediate from (4.8). Hence we can extend Theorem 4.6 to the case  $z = 1$ , in particular the function  $q \mapsto \zeta(q, 1)Z(q, 1)$  turns out to be analytic for  $\operatorname{Re}(q) > 1$ .

## References

- [1] T. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.
- [2] T. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Mathematics 41, Springer, New York, 1976.
- [3] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, Advanced Series in Nonlinear Dynamics 16, World Scientific, River Edge, 2000.
- [4] F.P. Boca, *Products of matrices  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  and the distribution of reduced quadratic irrationals*, J. Reine Angew. Math. **606** (2007) 149–165.
- [5] C. Bonanno, S. Graffi, S. Isola, *Spectral analysis of transfer operators associated to Farey fractions*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **19** (2008) 1–23.
- [6] C-H. Chang, D.H. Mayer, *The period function of the nonholomorphic Eisenstein series for  $\mathrm{PSL}(2, \mathbb{Z})$* , Math. Phys. Electron. J. **4** (1998), paper 6.
- [7] C-H. Chang, D.H. Mayer, *The transfer operator approach to Selberg’s zeta function and modular and Maass wave forms for  $\mathrm{PSL}(2, \mathbb{Z})$* , in A. Hejhal, M. Gutzwiller et al. eds, *Emerging applications of number theory*, 73–141, IMA Vol. Math. Appl., 109, Springer, New York, 1999.
- [8] M. Degli Esposti, S. Isola, A. Knauf, *Generalized Farey trees, transfer operators and phase transitions*, Comm. Math. Phys. **275** (2007) 297–329.
- [9] I. Efrat, *Dynamics of the continued fraction map and the spectral theory of  $SL(2, \mathbb{Z})$* , Invent. Math. **114** (1993), 207–218.
- [10] A. Erdély et al., *Higher transcendental functions*, Bateman manuscript project, vols. I–III, McGraw-Hill, New York, 1953–1955.
- [11] M. Giamperio, S. Isola, *A one-parameter family of analytic Markov maps with an intermittency transition*, Discrete Contin. Dyn. Syst. **12** (2005) 115–136.
- [12] A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France **84** (1956) 319–384.
- [13] S. Isola, *On the spectrum of Farey and Gauss maps*, Nonlinearity **15** (2002) 1521–1539.
- [14] J. Kallies, A. Özlük, M. Peter, C. Snyder, *On asymptotic properties of a number theoretic function arising out of a spin chain model in statistical mechanics*, Commun. Math. Phys. **222** (2001) 9–43.
- [15] J.C. Lagarias, *Number theory zeta functions and dynamical zeta functions*, in: T. Branson (Ed.), *Spectral Problems in Geometry and Arithmetic*, Contemp. Math. Vol. 237, Amer. Math. Soc., Providence, RI, 1999, pp. 45–86.
- [16] J.B. Lewis, *Spaces of holomorphic functions equivalent to even Maass cusp forms*, Invent. Math. **127** (1997) 271–306.

- [17] J.B. Lewis, D. Zagier, *Period functions for Maass wave forms. I*, Ann. Math. **153** (2001) 191–258.
- [18] D.H. Mayer, *On the thermodynamic formalism for the Gauss map*, Commun. Math. Phys. **130** (1990) 311–333.
- [19] D.H. Mayer, *Continued fractions and related transformations*, in: T. Bedford, M. Keane, C. Series (Ed.), Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces, Oxford University Press, New York, 1991, pp. 175–222.
- [20] D.H. Mayer, *The thermodynamic formalism approach to Selberg’s zeta function for  $PSL(2\mathbb{Z})$* , Bull. Amer. Math. Soc. **25** (1991) 55–60.
- [21] T. Prellberg, *Towards a complete determination of the spectrum of a transfer operator associated with intermittency*, J. Phys. A **36** (2003) 2455–2461.
- [22] T. Prellberg, J. Slawny, *Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transitions*, J. Stat. Phys. **66** (1992) 503–514.
- [23] H.H. Rugh, *Intermittency and regularized Fredholm determinants*, Invent. math. **135** (1999) 1–24.
- [24] D. Ruelle, *Zeta-functions for expanding maps and Anosov flows*, Invent. math. **34** (1976) 231–242.
- [25] C. Series, *The modular surface and continued fractions*, J. London. Math. Soc. **31** (1985) 69–80.